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RAY'S ALGEBRA,

PARTS FIRST AND SECOND:

CONTAINING

STATEMENTS AND SOLUTIONS OF QUESTIONS,

WITH REMARKS AND NOTES.

ALSO,

AN APPENDIX, CONTAINING INDETERMINATE AND DIOPHANTINE ANALYSIS, PROPERTIES OF NUMBERS, AND SCALES OF NOTATION.

BY

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PROFESSOR OF MATHEMATICS IN WOODWARD SCHOOL.

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KEY

TO

RAY'S ALGEBRA, PART FIRST.

☞ The numbers in parentheses, as seen in the margin of this KEY, refer to the corresponding number of example, under the same article in the Algebra.

INTELLECTUAL EXERCISES.

TO TEACHERS. As these exercises, except, perhaps, a few in Lesson XIV, can be readily solved without the aid of Algebra, by pupils having a good knowledge of Mental Arithmetic, it is unnecessary to occupy space with their solution.

Some instructors who use the Algebra, pay no attention to the intellectual exercises, but permit their pupils to begin with the preliminary definitions and principles. This course is proper with pupils of considerable maturity of mind, and who possess a good knowledge of arithmetic ; but in the case of learners generally, and especially those who are young, the intellectual exercises should be thoroughly studied.

Lesson 14.

(33) Let $x=$ the 1st, then $\frac{x}{3}=$ the 2d, and $\frac{x}{3}+2x=$ the 3d.

Then, $x+\frac{x}{3}+\frac{x}{3}+2x=44,$

adding, $3x+\frac{2x}{3}=44,$

$$\frac{11x}{3}=44,$$

$$\frac{x}{3}=4,$$

$$x=12.$$

the numbers are 12, 4, and 28.

(34) Let $x=$ the distance from A to B; then $\frac{x}{5}=$ dist. from B to C; and $x+\frac{x}{5}$ dist. from A to C.

Whence, $2x+\frac{2x}{5}=$ dist. from C to D;

and, $x+\frac{x}{5}+2x+\frac{2x}{5}=72,$

or, $3x+\frac{3x}{5}=72,$

$$\frac{18x}{5}=72,$$

$$\frac{x}{5}=4,$$

$x=20$, the distance from A to B.

$\frac{x}{5}=4$, the distance from B to C.

$2x+\frac{2x}{5}=48$, the distance from C to D.

(35) Let $x=$the number.

Then, $x+\frac{x}{2}+\frac{x}{4}+26=5x,$

or, $x+\frac{3x}{4}+26=5x,$

$$\frac{7x}{4}+26=\frac{20x}{4},$$

$$26=\frac{13x}{4},$$

$$2=\frac{x}{4},$$

or $x=8$.

(36) Let $x=$length of the body, then $\frac{x}{2}+6=$ length of tail.

Then, $x=\frac{x}{2}+6+6,$

or, $\frac{x}{2}=12,$

$x=24$, the length of the body;

$\frac{x}{2}+6=18$, the length of the tail;

and, $6+24+18=48$, the length of the whole fish.

(37) Let $x=$ his age.

then, $x+\frac{x}{2}+\frac{x}{3}+28=3x,$

or, $x+\frac{5x}{6}+28=3x,$

$$\frac{11x}{6}+28=\frac{18x}{6},$$

$$28=\frac{7x}{6},$$

$4=\frac{x}{6}$, and $x=24$, his age

ADDITION AND SUBTRACTION.

REMARK. Pupils sometimes experience difficulty from not attending to the definition of similar quantities; for example, by regarding such terms as $2a^2b$ and $3ab^2$ as similar. By attending to this point, and being careful to write similar terms under each other, no difficulty need be experienced in solving all the questions, in either addition or subtraction

---

MULTIPLICATION.

REMARKS. In algebraic multiplication, there are a few things which, although they affect no principle, are of sufficient importance to claim the pupil's attention.

1st. In multiplying two monomials together, it is customary to write the sign first, then the numeral coëfficient, and then the letters of the product from the left toward the right. Thus, in finding the product of $-2a^2$ by $+3ac$, we first write the sign of the product —, then 6, then a^3, and then c, making the whole $-6a^3c$. This is more convenient than writing the letters in the reverse order, because it corresponds to the manner of writing words.

2d. In multiplying by a polynomial, it is customary to multiply, first by the left term of the multiplier, next by the second term from the left, and so on. Although the result would evidently be the same if the operation were performed in a reverse order, yet this method is now so well established, that a different one would be regarded as unscholarly.

When a pupil understands addition of algebraic quantities, and how to multiply one monomial by another, he can encounter no real difficulty in performing any of the operations in multiplication. It is not, therefore, deemed necessary to insert the work of any of the examples.

DIVISION.

REMARKS. For reasons similar to those given under multiplication, it is customary, in dividing one monomial by another, to write, first the sign of the quotient, then the numeral coëfficient, if any, and then the literal part from left to right.

In dividing one polynomial by another, in order to conform to the general method of proceeding from the left toward the right, it is customary to divide the first term of the dividend by the first term of the divisor; this, however, affects no principle, as the division may be commenced at the right hand, by dividing the last term of the dividend by the last term of the divisor.

The divisor may be written either on the left or the right of the dividend; the latter is the French method, and is more convenient, because the quotient being written beneath, the quantities to be multiplied together before making each subtraction, are the most conveniently situated with regard to each other.

We here present the operation of a few of the more difficult examples, with which, and similar ones, pupils sometimes find difficulty.

Article 79.

NOTE. The terms in examples 12 and 17, require to be arranged; after which, the operations present no difficulty.

(14)

$$\begin{array}{l|l}
4a^4-5a^2x^2+x^4 & 2a^2-3ax+x^2 \\
\underline{4a^4-6a^3x+2a^2x^2} & (2a^2+3ax+x^2 \\
\quad 6a^3x-7a^2x^2 & \quad Ans. \\
\quad \underline{6a^3x-9a^2x^2+3ax^3} & \\
\qquad +2a^2x^2-3ax^3+x^4 & \\
\qquad \underline{2a^2x^2-3ax^3+x^4} &
\end{array}$$

(15)

$$\begin{array}{l|l}
x^4-y^4 & x-y \\
\underline{x^4-x^3y} & (x^3+x^2y+xy^2+y^3 \\
\quad +x^3y & \quad Ans. \\
\quad \underline{x^3y-x^2y^2} & \\
\qquad +x^2y^2 & \\
\qquad \underline{x^2y^2-xy^3} & \\
\qquad\quad +xy^3-y^4 & \\
\qquad\quad \underline{xy^3-y^4} &
\end{array}$$

(18)

$$\begin{array}{l|l}
4x^4-64 & 2x-4 \\
\underline{4x^4-8x^3} & (2x^3+4x^2+8x+16 \\
\quad +8x^3 & \quad Ans. \\
\quad \underline{8x^3-16x^2} & \\
\qquad +16x^2 & \\
\qquad \underline{16x^2-32x} & \\
\qquad\quad +32x-64 & \\
\qquad\quad \underline{32x-64} &
\end{array}$$

(21)

$$\begin{array}{l|l}
y^3+1 & y+1 \\
\underline{y^3+y^2} & (y^2-y+1 \\
\quad -y^2 & \quad Ans. \\
\quad \underline{-y^2-y} & \\
\qquad +y+1 & \\
\qquad \underline{y+1} &
\end{array}$$

$$
\begin{array}{l}
(24)\quad x^6-3x^4y^2+3x^2y^4-y^6 \;\; \underline{|x^3-3x^2y+3xy^2-y^3} \\
\underline{x^6-3x^5y+3x^4y^2-x^3y^3} \quad (x^3+3x^2y+3xy^2+y^3 \quad \textit{Ans.} \\
+3x^5y-6x^4y^2+x^3y^3+3x^2y^4 \\
\underline{3x^5y-9x^4y^2+9x^3y^3-3x^2y^4} \\
+3x^4y^2-8x^3y^3+6x^2y^4 \\
\underline{3x^4y^2-9x^3y^3+9x^2y^4-3xy^5} \\
+x^3y^3-3x^2y^4+3xy^5-y^6 \\
\underline{x^3y^3-3x^2y^4+3xy^5-y^6}
\end{array}
$$

FACTORING.

REMARKS. In solving the examples in factoring at the blackboard, the pupil should always explain why the given quantity can be separated into factors. Thus, $4a^2x^4-9b^2y^6=(2ax^2+3by^3)(2ax^2-3by^3)$, because it is the difference of the squares of two monomials, $2ax^2$ and $3by^3$. Again, x^3+1 can be separated into two factors, because it is the sum of the odd powers of two quantities x and 1, (Art. 94. 5th); and one of the factors is $x+1$.

It is shown in Art. 215, that the *direct* method of resolving a quadratic trinomial into its factors, is to place it equal to zero, and then find the roots of the equation; yet as the *indirect* method explained in Art. 95, presents no difficulty to an intelligent pupil, and is much shorter than the direct method, it should always be taught. Let it be kept distinctly before the mind of the pupil, that the whole difficulty consists in finding two numbers whose *sum* is equal to the coëfficient of the second term, and whose *product* is equal to the third term. Thus, in example 1, "What two numbers are those whose sum is 5, and product 6?" Any intelligent pupil will soon discover that 2 and 3 are the numbers required.

We here present the solution of the examples in

Article 95.

(2) $a^2+7a+12=(a+3)(a+4)$; because $+3+4=7$, and $3\times4=12$.

(3) $x^2-5x+6=(x-2)(x-3)$; because -2 and $-3=-5$, and $-2\times-3=+6$.

(4) $x^2-9x+20=(x-4)(x-5)$; because -4 and $-5=-9$, and $-4\times-5=+20$.

(5) $x^2+x-6=(x+3)(x-2)$; because $-2+3=+1$, and $-2\times3=-6$.

(6) $x^2-x-6=(x-3)(x+2)$; because $-3+2=-1$, and $-3\times2=-6$.

(7) $x^2+x-2=(x+2)(x-1)$; because $+2-1=+1$, and $-1\times 2=-2$.

(8) $x^2-13x+40=(x-5)(x-8)$; because -5 and $-8=-13$, and $-5\times -8=40$.

(9) $x^2-7x-8=(x-8)(x+1)$; because $-8+1=-7$, and $-8\times 1=-8$.

(10) $x^2+7x-18=(x-8)(x-2)$; because $-2+9=+7$, and $-2\times 9=-18$.

(11) $x^2-x-30=(x-6)(x+5)$; because $-6+5=-1$, and $-6\times 5=-30$.

(12) $3x^2+12x-15=3(x^2+4x-5)=3(x+5)(x-1.)$

(13) $a^2x^2-9a^2x+14a^2=a^2(x^2-9x+14)=a^2(x-7)(x-2)$.

(14) $2abx^2-14abx-60ab=2ab(x^2-7x-30)=2ab(x-10)(x+3)$.

(15) $2x^3-4x^2-30x=2x(x^2-2x-15)=2x(x-5)(x+3)$.

Article 96.

NOTE. In performing the operations on the slate or black-board, a line should be drawn across each canceled factor. We have not the means, except in the case of figures, of representing this by type. Thus, in example 4, following, a line should be drawn across each of the c's, and also across $(a-b)$ in the numerator and denominator

(2) $$\frac{(x-3)(x^2-1)}{x-1}=\frac{(x-3)(x+1)(x-1)}{(x-1)}=(x-3)(x+1)$$
$$=x^2-2x-3.$$

(3) $$\frac{(z^3+1)(z^2-1)}{(z+1)}=\frac{(z^3+1)(z-1)(z+1)}{z+1}=(z^3+1)(z-1)$$
$$=z^4-z^3+z-1.$$

(4) $$\frac{6a^2c-12abc+6b^2c}{2ac-2bc}=\frac{\not{6}c(a-b)(a-b)}{\not{2}c(a-b)}=3(a-b).$$

(5) $$\frac{(6ax+9ay)(4x^2-9y^2)}{4x^2+12xy+9y^2}=\frac{3a(2x+3y)(2x+3y)(2x-3y)}{(2x+3y)(2x+3y)}$$
$$=3a(2x-3y).$$

(6) $$\frac{(x^2-5x+6)(x^2-7x+12)}{x^2-6x+9}=\frac{(x-2)(x-3)(x-3)(x-4)}{(x-3)(x-3)}$$
$$=(x-2)(x-4).$$

GREATEST COMMON DIVISOR.

NOTE. All the examples for exercise, Art. 106, may be solved by merely separating the quantities into their factors, by the rules for factoring, Arts. 94; 95. But as the application of the direct rule for finding the greatest common divisor of two polynomials, is generally regarded by pupils as a difficult operation, we here present the solutions of all the examples.

Article 106.

(5) $5a^2+5ax=5a(a+x)$

By omitting the factor $5a$, (see Note 2), and dividing a^2-x^2 by the other factor $a+x$, we find there is no remainder; therefore $a+x$ is the *g. c. d.*

(6) $x^3-a^2x=x(x^2-a^2)$

$$\begin{array}{l|l} x^3-a^3 & x^2-a^2 \\ \underline{x^3-a^2x} & (x \\ a^2x-a^3 & \\ a^2(x-a) & \end{array}$$

$$\begin{array}{l|l} x^2-a^2 & x-a \quad g.\,c.\,d. \\ \underline{x^2-ax} & (x+a \\ ax-a^2 & \\ \underline{ax-a^2} & \end{array}$$

After dividing we find the first remainder contains a factor a^2 not contained in x^2-a^2, hence it is not a factor of the greatest common divisor, and should be omitted. See Note 3.

(7) $x^3-c^2x=x(x^2-c^2)$

$$\begin{array}{l|l} x^2+2cx+c^2 & x^2-c^2 \\ \underline{x^2-c^2} & (1 \\ 2cx+2c^2 & \\ 2c(x+c) & \end{array}$$

$$\begin{array}{l|l} x^2-c^2 & x+c \quad g.\,c.\,d. \\ \underline{x^2+cx} & (x-c \\ -cx-c^2 & \\ \underline{-cx-c^2} & \end{array}$$

(8)

$$\begin{array}{l|l} x^2+5x+6 & x^2+2x-3 \\ \underline{x^2+2x-3} & (1 \\ 3x+9 & \\ 3(x+3) & \end{array}$$

$$\begin{array}{l|l} x^2+2x-3 & x+3 \quad g.\,c.\,d. \\ \underline{x^2+3x} & (x-1 \\ -x-3 & \\ \underline{-x-3} & \end{array}$$

(9) $6a^2+11ax+3x^2 \mid 6a^2+7ax-3x^2$

$6a^2+7ax-3x^2 \quad (1$

$4ax+6x^2$; by factoring this we get

$2x(2a+3x)$

$6a^2+7ax-3x^2 \mid 2a+3x$

By completing this division, we find there is no remainder hence, $2a+3x$ is the greatest common divisor.

(10) $a^4-x^4 \mid a^3+a^2x-ax^2-x^3$

$a^4+a^3x-a^2x^2-ax^3 \quad (a-x$

$-a^3x+a^2x^2+ax^3-x^4$

$-a^3x-a^2x^2+ax^3+x^4$

$2a^2x^2-2x^4$; then by factoring

$2x^2(a^2-x^2)$

$a^3+a^2x-ax^2-x^3 \mid a^2-x^2$

By completing the division, we find there is no remainder, hence, a^2-x^2 is the greatest common divisor.

(11) $a^3-a^2x+3ax^2-3x^3 \mid a^2-5ax+4x^2$

$a^3-5a^2x+4ax^2 \quad (a+4x$

$+4a^2x-ax^2-3x^3$

$4a^2x-20ax^2+16x^3$

$+19ax^2-19x^3$; by factoring this we get

$19x^2(a-x)$

By dividing $a^2-5ax+4x^2$ by $a-x$, we find there is no remainder; hence, $a-x$ is the greatest common divisor.

(12) $a^2x^4-a^2y^4=a^2(x^4-y^4) : x^5+x^3y^2=x^3(x^2+y^2)$.

By the principle of Note 3, neither of the factors a^2 or x^3, can form factors of the greatest common divisor; then by dividing x^4-y^4 by x^2+y^2, we find there is no remainder; hence, the latter quantity is the required greatest common divisor.

(13) $a^{13}-x^{13} \mid a^5-x^5$

$a^{13}-a^8x^5 \quad (a^8+a^3x^5$

$+a^8x^5-x^{13}$

$a^8x^5-a^3x^{10}$

$+a^3x^{10}-x^{13}$

$x^{10}(a^3-x^3)$

$a^5-x^5 \mid a^3-x^3$

$a^5-a^2x^3 \quad (a^2$

$a^2x^3-x^5$

$x^3(a^2-x^2)$

$a^3-x^3 \mid a^2-x^2$

$a^3-ax^2 \quad (a$

$+ax^2-x^3$

$x^2(a-x)$

By dividing a^2-x^2 by $a-x$, we find there is no remainder; hence, $a-x$ is the greatest common divisor sought.

LEAST COMMON MULTIPLE.

NOTE. The pupil should be reminded, that the operation of finding the least common multiple in algebra, involves precisely the same principles as in arithmetic.

(6)

$a-x$	$4a^2(a-x)$	$6ax^4(a^2-x^2)$
$2a$	$4a^2$	$6ax^4(a+x)$
	$2a$	$3x^4(a+x)$

$(a-x)\times 2a\times 2a\times 3x^4(a+x)=12a^2x^4(a^2-x^2)$. *Ans.*

(7)

2	$8x^2(x-y)$	$3a^4x^2$	$12axy^2$
2	$4x^2(x-y)$	$3a^4x^2$	$6axy^2$
$3a$	$2x^2(x-y)$	$3a^4x^2$	$3axy^2$
x	$2x^2(x-y)$	a^3x^2	xy^2
x	$2x(x-y)$	a^3x	y^2
	$2(x-y)$	a^3	y^2

$2\times 2\times 3a\times x\times x\times 2(x-y)\times a^3\times y^2=24a^4x^2(x-y)$. *Ans.*

(8)

$x-y$	$10a^2x^2(x-y)$	$15x^5(x+y)$	$12(x^2-y^2)$
$x+y$	$10a^2x^2$	$15x^5(x+y)$	$12(x+y)$
2	$10a^2x^2$	$15x^5$	12
3	$5a^2x^2$	$15x^5$	6
$5x^2$	$5a^2x^2$	$5x^5$	2
	a^2	x^3	2

$(x-y)(x+y)\times 2\times 3\times 5x^2\times a^2\times x^3\times 2=60a^2x^5(x^2-y^2)$. *Ans.*

ALGEBRAIC FRACTIONS.

REMARK. The pupil can experience but little difficulty in solving any of the examples in fractions, if he is well acquainted with the fundamental operations, and Factoring. We here present solutions of the only examples likely to occasion difficulty.

CASE 1. *To reduce a fraction to its lowest terms.*

Article 128.

(14) $\frac{3z^3-24z+9}{4z^3-32z+12}=\frac{3(z^3-8z+3)}{4(z^3-8z+3)}=\frac{3}{4}$. *Ans*

(15) $\frac{5a^2+5ax}{a^2-x^2}=\frac{5a(a+x)}{(a+x)(a-x)}=\frac{5a}{a-x}$. *Ans.*

(16) $\frac{n^2-2n+1}{n^2-1}=\frac{(n-1)(n-1)}{(n+1)(n-1)}=\frac{n-1}{n+1}$. *Ans.*

(18) $\frac{x^3-xy^2}{x^4-y^4}=\frac{x(x^2-y^2)}{(x^2+y^2)(x^2-y^2)}=\frac{x}{x^2+y^2}$. *Ans*

(19) $\frac{a^2+b^2}{a^4-b^4}=\frac{a^2+b^2}{(a^2+b^2)(a^2-b^2)}=\frac{1}{a^2-b^2}$. *Ans.*

(20) $\frac{x^2-y^2}{x^2-2xy+y^2}=\frac{(x+y)(x-y)}{(x-y)(x-y)}=\frac{x+y}{x-y}$. *Ans*

(21) $\frac{x^3-ax^2}{x^2-2ax+a^2}=\frac{x^2(x-a)}{(x-a)(x-a)}=\frac{x^2}{x-a}$. *Ans.*

(22) $\frac{2x^2-6x}{x^2-x-6}=\frac{2x(x-3)}{(x+2)(x-3)}=\frac{2x}{x+2}$. *Ans.*

(23) $\frac{x^2+2x-15}{x^2+8x+15}=\frac{(x+5)(x-3)}{(x+5)(x+3)}=\frac{x-3}{x+3}$. *Ans.*

Article 129.

(7) $\frac{x^3y^2+x^2y^3}{ax^2y+axy^2}=\frac{xy(x^2y+xy^2)}{a(x^2y+xy^2)}=\frac{xy}{a}$. *Ans.*

(8) $\frac{4a+4b}{2a^2-2b^2}=\frac{4(a+b)}{2(a+b)(a-b)}=\frac{2}{a-b}$. *Ans.*

(9) $\frac{n^3-2n^2}{n^2-4n+4}=\frac{n^2(n-2)}{(n-2)(n-2)}=\frac{n^2}{n-2}$. *Ans.*

(10) $\frac{x^2+2x-3}{x^2+5x+6}=\frac{(x+3)(x-1)}{(x+3)(x+2)}=\frac{x-1}{x+2}$. *Ans.*

MULTIPLICATION AND DIVISION OF FRACTIONS.

REMARK. The only difficulty in solving any of the examples in either multiplication or division of fractions, consists in reducing the resulting fraction to its lowest terms. The difficulty may be avoided, generally, by first indicating the operations to be performed, then factoring, and then canceling the factors common to both terms. We here present the solution of a few of the examples, both in multiplication and division

Article 140.

(11) $\frac{x}{a+x}\times\frac{a^2-x^2}{x^2}\times\frac{a}{a-x}=\frac{x(a+x)(a-x)a}{(a+x)x^2(a-x)}=\frac{a}{x}$. *Ans.*

(12) $\frac{x^2+y^2}{x-y}\times\frac{x^2-y^2}{x+y}\times\frac{a}{1}=\frac{(x^2+y^2)(x+y)(x-y)a}{(x-y)(x+y)}=a(x^2+y^2)$. *A*

(16) $c+\frac{cx}{c-x}=\frac{c^2}{c-x}$. $\frac{c^2}{c-x}\times\frac{c^2-x^2}{x+1}=\frac{c^2(c+x)(c-x)}{(c-x)(x+1)}=\frac{c^2(c+x)}{x+1}$.

Article 141.

(22) $\frac{a^3+b^3}{2+3x}\div(ab+b^2)=\frac{a^3+b^3}{2+3x}\times\frac{1}{ab+b^2}=\frac{(a+b)(a^2-ab+b^2)}{(2+3x)b(a+b)}=$
$\frac{a^2-ab+b^2}{2b+3bx}$. *Ans.*

(23) $\frac{x^2-y^2}{3a}\div(x^2-xy)=\frac{x^2-y^2}{3a}\times\frac{1}{x^2-xy}=\frac{(x+y)(x-y)}{3ax(x-y)}=\frac{x+y}{3ax}$

Article 142.

(21) $\frac{a-b}{a+b}\div\frac{a^2-b^2}{a^2+2ab+b^2}=\frac{a-b}{a+b}\times\frac{a^2+2ab+b^2}{a^2-b^2}=$
$\frac{(a-b)(a+b)(a+b)}{(a+b)(a+b)(a-b)}=1$. *Ans*

(23) $\frac{2x^2}{a^3+x^3}\div\frac{x}{a+x}=\frac{2x^2}{a^3+x^3}\times\frac{a+x}{x}=\frac{2x^2(a+x)}{(a+x)(a^2-ax+x^2)x}=$
$\frac{2x}{a^2-ax+x^2}$ *Ans*

Article 144.

RESOLUTION OF FRACTIONS INTO SERIES.

(2)

$$\begin{array}{l|l} 1 & 1+x \\ \underline{1+x} & (1-x+x^2-\&c. \\ -x & \\ \underline{-x-x^2} & \\ +x^2 & \end{array}$$

(3)

$$\begin{array}{l|l} ax & a-x \\ \underline{ax-x^2} & (x+\frac{x^2}{a}+\frac{x^3}{a^2}+\&c. \\ +x^2 & \\ \underline{+x^2-\frac{x^3}{a}} & \\ +\frac{x^3}{a} & \\ \underline{+\frac{x^3}{a}-\frac{x^4}{a^2}} & \end{array}$$

(4)

$$\begin{array}{l|l} 1+x & 1-x \\ \underline{1-x} & (1+2x+2x^2+\&c. \\ +2x & \\ \underline{+2x-2x^2} & \\ +2x^2 & \\ \underline{+2x^2-2x^3} & \\ +2x^3 & \end{array}$$

(6)

$$\begin{array}{l|l} x+2 & x+1 \\ \underline{x+1} & (1+\frac{1}{x}-\frac{1}{x^2}+\&c. \\ 1 & \\ \underline{1+\frac{1}{x}} & \\ -\frac{1}{x} & \\ \underline{-\frac{1}{x}-\frac{1}{x^2}} & \end{array}$$

EQUATIONS OF THE FIRST DEGREE, OR SIMPLE EQUATIONS.

REMARK. The only difficulty pupils will be likely to experience in solving the examples in Articles 154 and 155, will be where a fraction whose numerator contains two or more terms, is preceded by the sign minus, as in example 10, Art. 154, or in examples 16, 17, &c., Art. 155. This may be obviated by the pupil writing the numerator of the fraction in a vinculum when the equation is cleared of fractions, and then proceeding to perform the operations indicated. It will thus be seen, that the effect of the minus sign before a fraction, is, to change the sign of each term of the numerator. (See Art. 132). For illustration, take example 16, Art. 155.

$$\frac{x+2}{3}-\frac{x-3}{4}=x-2-\frac{x-1}{2}.$$

Multiplying both sides by 12, to remove the denominators,

$$4(x+2)-3(x-3)=12x-24-6(x-1),$$

or, $4x+8-3x+9=12x-24-6x+6,$

by transposing, $4x-3x-12x+6x=-24+6-8-9$;

by reducing, $-5x=-35$;

$$x=7.$$

QUESTIONS PRODUCING EQUATIONS OF THE FIRST DEGREE.

Article 156.

(16) Let $x=$A's share, then $2x=$ B's,
and $x+2x=42$.
Whence $x=14$.

(17) Let $x=$ the first part, then $2x=$ the second, and $3x=$ the third, and $x+2x+3x=48$;
from which $x=8$.

(18) Let $x=$the first part, then $3x=$ the second, and $3x\times2=6x=$ the third part.
Therefore, $x+3x+6x=60$;
from which $x=6$.

(19) Let $x=$ the number of each,
then $1x$ or $x=$ cost of the apples,
$2x=$ " " lemons,
and $5x=$ " " oranges.
Therefore, $x+2x+5x=56$;
from which $x=7$.

(20) Let $x=$ cost of an apple, then $2x$ cost of a lemon.
$5x=$ cost of all the apples, and $3\times2x=$ cost of all the lemons.
Therefore, $5x+6x=22$;
from which $x=2$.

(21) Let $x=$ C's age, then $2x=$ B's age, and $4x=$ A's age.
Therefore, $x+2x+4x=98$;
from which $x=14$.

(22) Let $x=$ A's cents, then $3x=$ B's, $x+\frac{3x}{3}=2x=$ C's, and $3x+2x=5x=$ D's.
Therefore, $x+3x+2x+5x=44$;
from which $x=4$.

(23) Let $x=$ age of youngest, then $2x=$ common difference of their ages, and $3x=$ age of second, $5x=$ age of third, and $7x=$ age of fourth.
Therefore, $x+3x+5x+7x=48$;
from which $x=3$.

(25) Let $5x$ and $7x$ represent the numbers, since $5x$ is to $7x$ as 5 to 7.
Then $5x+7x=60$;
from which $x=5$.
Hence, $5x=25$, and $7x=35$.

(26) Let $2x$, $3x$, and $5x$ represent the parts; then
$2x+3x+5x=60$;
from which $x=6$.
Hence, $2x=12$, $3x=18$, and $5x=30$.

(27) Let $3x$, $5x$, $7x$, and $8x$ represent the parts; then
$3x+5x+7x+8x=92$;
from which $x=4$.
Hence, $3x=12$, $5x=20$, $7x=28$, and $8x=32$.

(28) Let $2x$, $3x$, and $5x$ represent the parts; these will evidently fulfill the second condition, since $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{5}$ of the third, are each equal to x
Then $2x+3x+5x=60$;
from which $x=6$.
Hence, $2x=12$, $3x=18$, and $5x=30$.

(29) Let $x=$ the number.
Then $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=65$;
from which $x=60$.
Or, let $12x=$ the number; then
$6x+4x+3x=65$;
from which $x=5$; and $12x=60$.
To avoid fractions, we choose $12x$, because it is a multiple of 2, 3, and 4.

(30) Let $x=$ the number.

Then $\frac{x}{5}-\frac{x}{7}=4$;

from which $x=70$.

By putting $35x$ for the number, we may avcid fractions.

(31) Let $x=$ A's age, then $2\frac{4}{5}x=$ B's.

Therefore, $x+\frac{14x}{5}=76$;

or, $\frac{5x}{5}+\frac{14x}{5}=\frac{19x}{5}=76$;

by clearing of fractions, $19x=76\times5$,

by dividing by 19, $x=4\times5=20$, A's age.

(32) Let $x=$ A's part, then $\frac{2x}{3}=$ B's, and $\frac{3x}{7}=$C's.

Therefore, $x+\frac{2x}{3}+\frac{3x}{7}=88$;

from which $x=42$.

(33) Let $x=$ C's share, then $\frac{3x}{4}=$ B's, and $\frac{3}{5}$ of $\frac{3x}{4}=\frac{9x}{20}=$ A's.

Therefore, $\frac{9x}{20}+\frac{3x}{4}+x=440$;

from which $x=200$.

(34) Let $3x=$ distance from A to B, then $5x=$ distance from B to C;

also, $\frac{3x}{3}+5x=6x$, and $\frac{6x}{3}=2x=$ distance from C to D.

Therefore, $3x+5x+2x=120$;

from which $x=12$.

Hence, $3x=36$, and $5x=60$.

(35) Let $3x=$ capital

Then $3x-\frac{3x}{3}=2x=$ capital at close of 1st year;

$2x+\frac{2}{5}$ of $2x=2x+\frac{4x}{5}=\frac{14x}{5}=$ cap. close 2nd year;

$\frac{14x}{5}-\frac{1}{7}$ of $\frac{14x}{5}=\frac{14x}{5}-\frac{2x}{5}=\frac{12x}{5}=$ cap. 3d year.

Therefore, $\frac{12x}{5}=1236$;

from which $3x=1545$.

(36) Let $x=$ rent last year.

Then $x+\frac{5x}{100}=168$;

from which $x=160$.

(37) Let $x=$ the less part, then $x+6=$ the greater.

Therefore, $x+x+6=32$;

from which $x=13$.

(38) Let $x=$ votes of unsuccessful candidate,

Then $x+50=$ votes of successful candidate.

Therefore, $x+x+50=256$;

from which $x=103$.

(39) Let $x=$ A's,

then $x+100=$ B's,

and $x+100+270=x+370=$ C's.

Therefore, $x+x+100+x+370=1520$;

from which $x=350$.

(40) Let $x=$ number of women,

then $x+4=$ " men,

and $2x+4+10=2x+14=$ number of children.

Therefore, $x+x+4+2x+14=90$;

from which $x=18$.

(41) Let $x=$ number of yards cut off,

then $x-9=$ number of yards remaining.

Therefore, $x+x-9=45$;

from which $x=27$.

(42) Let $x=$ the number.

Then $7x-20=20-x$;

from which $x=5$.

(43) Let $x=$ each daughter's share, then $2x=$ each son's share,

$3x=$ what all the daughters will receive;

$4x=$ " both the sons will receive;

then $7x-500=$ what the widow will receive.

Therefore, $3x+4x+7x-500=6500$;

from which $x=500$.

(44) Let $x=$ the number of days,

then $20x=$ distance 1st travels,

and $30x=$ " 2nd travels.

Therefore, $20x+30x=400$;

from which $x=8$.

(45) Let $x=$ the number of hours.
Then $3x+30=$ miles B travels,
and $5x=$ miles A travels.
Therefore, $5x=3x+30$.
from which $x=15$.

(46) Let $x=$ the number.
Then $\frac{x}{2}+\frac{x}{3}-44=\frac{x}{5}-6$;
from which $x=60$.

(47) Let $x=$ time past noon.
Then $12-x$ time to midnight.
Therefore, $x+\frac{x}{2}+\frac{x}{3}+\frac{2x}{5}=\frac{12-x}{6}$;
clearing of fractions and reducing, we find $72x=60$;
whence $x=\frac{5}{6}$ hr. $=50$ min.

(48) Let $x=$ one part, then $120-x=$ the other.
Therefore, $\frac{120-x}{x}=1\frac{1}{2}$ or $\frac{3}{2}$;
from which $x=48$.

(49) Let $x=$ the number.
Then $\frac{7x+3}{2}-4=15$;
from which, $x=5$.

(50) Let $x=$ the number.
Then $\frac{5x-24}{6}+13=x$;
from which $x=54$.

(51) Let $3x=$ A's capital, then $2x=$ B's.
Then $3x-100=$ A's after losing \$100;
$2x+100=$ B's after gaining \$100.
Therefore, $2x+100-\frac{5}{7}(3x-100)=134$;
from which $x=262$.
Hence, $3x=786$, and $2x=524$.

(52) Let $x=$ his money.
Then $x-\left(\frac{2x}{3}+3\right)=\frac{x}{5}+7$;
from which $x=75$.

(53) Let $5x=$ annual income of each.
Then $x=$ what A saves, and $4x=$ what he spends yearly;
also, $4x+25=$ what B spends yearly;
and $5x-(4x+25)=x-25=$ what B saves yearly
Therefore, $5(x-25)=200$;
from which $x=65$.
Hence, $5x=325$.

(54) Let $x=$ the number of pounds.

Then $\frac{2x}{3}+10=$ lbs. of nitre;

$\frac{2x}{23}+1=$ lbs. of sulphur;

$\frac{x}{3}-17=$ lbs. of charcoal;

Therefore, $\frac{2x}{x}+10+\frac{2x}{23}+1+\frac{x}{3}-17=x$.

By omitting $\frac{2x}{3}+\frac{x}{3}$ on the right, and its equivalent x on the left, and reducing,

we find $\frac{2x}{23}-6=0$;
whence $2x=6\times23$, and $x=3\times23=69$.

(55) Let $x=$ cost of harness, then $3x=$ cost of horse;
$4x\times2\frac{2}{3}=\frac{32x}{3}$, therefore $\frac{32x}{3}-19=$ cost of chaise.

Hence, $x+3x+\frac{32x}{3}-19=245$;

transposing and reducing, $\frac{44x}{3}=264$;
whence $44x=264\times3$,
and $x=6\times3=18$.

(56) Let $3x$ and $4x$ represent the number.
Then $3x+4 : 4x+4 :: 5 : 6$,
whence $6(3x+4)=5(4x+4)$;
from which $x=2$. Hence, $3x=6$, and $4x=8$.

(57) Let $2x$ and $5x$ represent the numbers.
Then $2x-2 : 5x-2 :: 3 : 8$,
whence $8(2x-2)=3(5x-2)$;
from which $x=10$.
Hence, $2x=20$, and $5x=50$.

(58) Let $x=$ the number of years.
Then $25+x : 30+x :: 8 : 9$;
whence $9(25+x)=8(30+x)$;
whence, by reducing, $x=15$.
Again, let $x=$ the number of years, since their ages were as 1 to 2.
Then $25-x : 30-x :: 1 : 2$;
whence $2(25-x)=30-x$,
and, by reducing, $x=20$.

(59) Let $x=$ the number of hours.
Then, since the first fills the cistern in $1\frac{1}{3}$ hours, it fills $\frac{1}{1\frac{1}{3}}=\frac{3}{4}$ of it in 1 hour, and in x hours it will fill $\frac{3x}{4}$ part of it.
In like manner, the second pipe fills $\frac{1}{3\frac{1}{3}}=\frac{3}{10}$ of the cistern in 1 hour, and in x hours it will fill $\frac{3x}{10}$ part of it.
Also, the third pipe fills $\frac{1}{5}$ in 1 hour, and in x hours will fill $\frac{x}{5}$ part of it.
Therefore, $\frac{3x}{4}+\frac{3x}{10}+\frac{x}{5}=1$, or the whole of the cistern;
whence $x=\frac{4}{5}$ hour $=48$ min.

(60) Let $x=$ the number of days.
Then, since the first does it in seven days, he does $\frac{1}{7}$ of it in 1 day, and in x days, $\frac{x}{7}$.
In like manner, the second does $\frac{1}{6}$ in 1 day, and in x days, $\frac{x}{6}$
The third does $\frac{1}{9}$ in 1 day, and in x days $\frac{x}{9}$.
Therefore, $\frac{x}{7}+\frac{x}{6}+\frac{x}{9}=1$, or the whole;
from which $x=2\frac{28}{53}$.

(61) Let $3x=$ money,
$3x-\frac{3x}{3}=2x$.
Then $2x+50-\frac{1}{10}(2x+50)+37=100$;
from which $x=10$.
Hence, $3x=30$.

(62) Let $5x=$ yearly salary;
$5x-\frac{2}{5}$ of $5x=3x$;
$3x-\frac{1}{3}$ of $3x=2x$.
Then $2x-\frac{2x}{5}=120$;
from which $x=75$, and $5x=375$.

(63) Let $x=$ value of suit of clothes.
Then $80+x=$ yearly wages;
and $\frac{80+x}{12}=$ monthly wages.
Therefore, $7\left(\frac{80+x}{12}\right)=x+35$,
from which $x=28$.

(64) Let $x=$ days it will last the woman.
Then, $\frac{1}{x}=$ part the woman can drink in 1 day;
since both can drink it in 6 days, they can drink $\frac{1}{6}$ of it in 1 day;
since the man can drink it in 10 days, he can drink $\frac{1}{10}$ of it in 1 day.
Therefore, $\frac{1}{6}-\frac{1}{10}=\frac{1}{x}$;
from which $x=15$.

(65) Let $x=$ the distance in miles.
Then $\frac{x}{15}=$ hours in going from C to L;
and $\frac{x}{10}=$ " in going from L to C.
Therefore, $\frac{x}{15}+\frac{x}{10}=25$;
from which $x=150$.

(66) Let $x=$ what B lost; then $2x=$ what A lost.
Therefore, $\frac{240-2x}{3}=96-x$;
from which $x=48$.

(67) Let $x=$ the whole number of gallons.
Then $\frac{x}{2}+25=$ gallons of wine;

and $\frac{x}{3}-5=$ gallons of water.

Therefore, $x=\frac{x}{2}+25+\frac{x}{3}-5$.

Whence $x=120$, and $\frac{x}{2}+25=85=$ gallons of wine;

and $\frac{x}{3}-5=35=$ gallons of water.

(68) Let $x=$ less part, then $91-x=$ the greater part,
and $91-2x=$ the difference of the parts.

Therefore, $\frac{91-x}{91-2x}=7$;

from which $x=42$.

(69) By representing the four parts by $x-2$, $x+2$, $\frac{1}{2}x$, and $2x$, we at once fulfill the last four conditions.
Therefore, $x-2+x+2+\frac{1}{2}x+2x=72$;
by adding, $4\frac{1}{2}x=72$,
whence $x=16$.
Then $x-2=14$; $x+2=18$; $\frac{1}{2}x=8$; and $2x=32$.

(70) Let $x=$ length of each piece.
Then $3(x-19)+x-17=142$;
from which $x=54$.

(71) Let $x=$ the number of sheep.

Then $\frac{x}{10}=$ acres ploughed;

and $\frac{x}{4}=$ acres of pasture.

Therefore, $\frac{x}{10}+\frac{x}{4}=161$;

from which $x=460$.

(72) Let $x=$ greater part, then $34-x=$ less part,
$18-(34-x)=x-16$;
Therefore, $x-18 : x-16 :: 2 : 3$.
Whence $3(x-18)=2(x-16)$;
from which $x=22$.

(73) Let $x=$ the number of beggars.
Then $3x-8=$ his money;
also, $2x+3=$ his money;
Therefore, $3x-8=2x+3$;
from which $x=11$.

(74) To avoid fractions, let $16x=$ the number of apples.

$16x$
$8x-8=$ number distributed to the first;
$8x+8=$ number left;
$4x+4-8=$ number " " " second;
$4x+12=$ number left;
$2x+6-8=$ number " " " third;
$2x+14=$ number left;
$x+7-8=$ number " " " fourth;
$x+15=$ number left.

Therefore, $x+15=20$;
from which $x=5$, and $16x=80$.
The question may be solved in the same manner by letting $x=$ the number of apples.

(75) Let $x=$ number of days, in which B alone could reap it.
Then $\frac{1}{x}=$ part B could reap in 1 day, and $\frac{6}{x}=$ the part he could reap in 6 days.
Since A can reap it in 20 days, he can reap $\frac{1}{20}$ in 1 day, and in 16 days, $\frac{16}{20}$.
Therefore, $\frac{16}{20}+\frac{6}{x}=1$, the whole;
from which $x=30$.

(76) Let $\frac{1}{2}x$ and $\frac{2}{3}x$ represent the numbers.
Then $\frac{1}{2}x+6 : \frac{2}{3}x+5 :: \frac{2}{5} : \frac{1}{2}$.
Whence $\frac{1}{2}(\frac{1}{2}x+6)=\frac{2}{5}(\frac{2}{3}x+5)$;
from which $x=60$; hence, $\frac{1}{2}x=30$, and $\frac{2}{3}x=40$.

(77) Let $x=$ price of a bushel of barley.
Then $\frac{4x+90}{9}=$ price of a bushel of oats;
therefore, $x+3 : \frac{4x+90}{9} :: 8 : 5$.
Whence $5(x+3)=8\left(\frac{4x+90}{9}\right)$;
from which $x=45$.

(78) Let $2x=$ distance from A to B,
then $3x=$ distance from C to D;
$\frac{2x}{4}+\frac{3x}{2}=2x=3$ times the distance from B to C;

therefore, $\frac{2x}{3}$= distance from B to C.

Hence, $2x+\frac{2x}{3}+3x=34$;

from which $x=6$; hence, $2x=12$; $3x=18$; and $\frac{2x}{3}=4$.

(79) Let x= the lbs. of rice,

then $\frac{x+5}{2}=\frac{1}{3}$ the weight of the flour, since $\frac{1}{2}$ of $\frac{2}{3}$ is $\frac{1}{3}$,

and $\frac{3x+15}{2}$= the weight of the flour;

$\frac{1}{5}\left(\frac{3x+15}{2}+x\right)=\frac{x+3}{2}$= weight of the water.

Therefore, $x+\frac{3x+15}{2}+\frac{x+3}{2}=15$;

from which $x=2$.

Article 161.

QUESTIONS PRODUCING EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

(5) Let x= the price of a lb. of coffee, and y= the price of a lb. of sugar.
Then $5x+3y=79$, (1),
and $3x+5y=73$, (2);
from which $x=11$, and $y=8$.

(6) Let x= the price of a horse, and y= the price of a cow.
Then $9x+7y=300$, (1),
and $6x+13y=300$, (2);
from which $x=24$, and $y=12$.

(7) Let x= the price of a dozen of port, and y= the price of a dozen of sherry.
Then $20x+30y=120$, (1),
and $30x+25y=140$, (2);
from which $x=3$, and $y=2$.

(8) Let x and y represent the numbers.
Then $\frac{1}{2}x+\frac{1}{3}y=22$, (1),
and $\frac{1}{4}x+\frac{1}{5}y=12$, (2);
from which $x=24$, and $y=30$.

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(9) Let x represent the greater, and y the less of the two numbers.

Then $x+\frac{1}{3}y=37$, (1),
and $y-\frac{1}{4}x=20$, (2);
from which $x=28$, and $y=27$.

(10) Let $x=$ the first, and $y=$ the second of the two number

Then $\frac{1}{2}x-\frac{1}{3}y=5$, (1),
and $\frac{1}{4}x-\frac{1}{5}y=2$, (2);
from which $x=20$, and $y=15$.

(11) Let $x=$ value of first horse, and $y=$ value of second.

Then $x+25=2y$, (1),
and $y+25=3x$, (2);
from which $x=15$, and $y=20$.

(12) Let $x=$ A's property, and $y=$ B's.

Then $x+50=y-20$, (1),
and $3x+5y=2350$, (2);
from which $x=250$, and $y=320$.

(13) Let $x=$ gallons 1st holds, and $y=$ gallons 2d holds.

Then $\frac{2x}{5}+96=\frac{3y}{4}$, (1),

and $\frac{5y}{8}=\frac{4x}{9}$, (2);

from which $x=720$, and $y=512$.

(14) Let $x=$ digit in ten's place, and $y=$ digit in unit's place.
Then $10x+y=$ the number, and $10y+x=$ number inverted,

therefore, $\frac{10x+y}{x+y}=7$, (1),

and $\frac{10y+x}{x+y+4}=3$, (2);

from which $x=8$, and $y=4$.

(15) Let $x=$ the numerator, and $y=$ the denominator of the fraction.

Then $\frac{x+8}{y}=2$. (1),

and $\frac{x}{y-5}=3$, (2);

from which $x=6$, and $y=7$.

(16) Let $x=$ A's age, and $y=$ B's.

Then $x+y+18=2x$, (1),
and $x-y-6=y$, (2);
from which $x=30$, and $y=12$.

(17) Let $x=$ the greater, and $y=$ the less of the two numbers.
Then $x+y=37$, (1),
and $\frac{4x-3y}{6}=6$, (2);
from which $x=21$, and $y=16$.

(18) Let $x=$ the numerator, and $y=$ the denominator of the fraction.
Then $\frac{x-3}{y-3}=\frac{1}{4}$, (1),
and $\frac{x+5}{y+5}=\frac{1}{2}$, (2);
from which $x=7$, and $y=19$.

(19) Let $x=$ sum given to A, and $y=$ sum given to B.
Then $x-\frac{1}{4}x=\frac{3x}{4}=$ A's capital at close of the year,
and $y+\frac{1}{4}y=\frac{5y}{4}=$B's capital at close of the year.
Then $x+y=2400$, (1),
and $\frac{3x}{4}=\frac{5y}{4}$, (2);
from which $x=1500$, and $y=900$.

(20) Let $x=$ the less, and $y=$ the greater of the two numbers
Then $y-1=4x$, (1),
and $x+3=\frac{1}{3}y$, (2);
from which $x=8$, and $y=33$.

(21) Let $x=$ A's, and $y=$ B's.
Then $x+100=y-100$, (1),
and $2(x-100)=y+100$, (2);
from which $x=500$, and B$=700$.

(22) Let $x=$ the greater, and $y=$ the less of the two numbers
Then $5x+7y=198$, (1),
and $\frac{x}{5}+\frac{y}{7}=6$, (2);
from which $x=20$, and $y=14$.

(23) Let $x=$ A's age, and $y=$B's.
Then $x-7$ and $y-7$, represent their ages 7 years ago;
and $x+7$ and $y+7$, represent their ages 7 years hence.
Therefore, $x-7=3(y-7)$, (1),
and $x+7=2(y+7)$, (2);
from which $x=49$, and $y=21$.

(24) Let $x=$ digit in ten's place, and $y=$ digit in unit's place

Then $10x+y=$ the number,

and $10y+x=$ the number inverted.

Therefore, $\frac{10x+y}{x+y}=4$, (1),

and $10x+y+27=10y+x$, (2);

from which $x=3$, and $y=6$.

(25) Let $x=$ value of a lb. of the first, and $y=$ value of a lb of the second

Then $x+y=20$, (1),

and $3x+5y=11(3+5)=88$, (2);

from which $x=6$, and $y=14$.

(26) Let $x=$ the number of lemons, and $y=$ the number of oranges.

Then $3x+5y=84$, (1);

since by selling $\frac{1}{2}$ of the lemons and $\frac{1}{3}$ of the oranges for 40 cents he cleared 8 cents;

therefore, $\frac{3x}{2}+\frac{5y}{3}=40-8=32$, (2);

from which $x=8$, and $y=12$.

(27) Let $x=$ number of peaches and $y=$ number of apples.

Then $\frac{x}{4}=$ cost of the peaches, and $\frac{y}{5}=$ cost of the apples;

therefore, $\frac{x}{4}+\frac{y}{5}=30$, (1),

$\frac{1}{2}$ of $\frac{x}{4}=\frac{x}{8}=$ cost of $\frac{1}{2}$ the peaches;

$\frac{1}{3}$ of $\frac{y}{5}=\frac{y}{15}=$ " " $\frac{1}{3}$ the apples;

therefore, $\frac{x}{8}+\frac{y}{15}=13$, (2);

from which $x=72$, and $y=60$.

(28) Let $x=$ A's money, and $y=$ B's.

Thus $x+\frac{1}{5}y=500$, (1),

and $y+\frac{1}{4}x=600$, (2);

from which $x=400$, and $y=500$.

(29) Let $x=$ number of yards in first piece, and $y=$ number of yards in 2d.

Then $4x+7y=236$, (1),
and $\frac{7}{8}(4x)+\frac{3}{5}(7y)=160+8$,
or $\frac{7x}{2}+\frac{21y}{5}=168$, (2);
from which $x=24$, and $y=20$.

(30) Let $x=$ the father's, and $y=$ the son's age.
Then $x-6=3\frac{1}{3}(y-6)$, (1),
and $x+3=2\frac{1}{6}(y+3)$, (2);
from which $x=36$, and $y=15$.

(31) Let $x=$ value of the first horse, and $y=$ value of the second.
Then $x+50=y+2+8$, (1);
$x+2=$ value of first horse with worst saddle,
$y+50=$ " " second " " best "
therefore, $y+50 : x+2 :: 15 : 4$,
whence, $4(y+50)=15(x+2)$, (2);
from which $x=30$, and $y=70$.

(32) Let $x=$ number of bushels of oats, and $y=$ number of bushels of rye.
Then, by the 1st condition, $x+6 : y+6 :: 7 : 6$,
whence $6(x+6)=7(y+6)$, (1).
By the 2d condition, $x-6 : y-6 :: 6 : 5$,
whence $5(x-6)=6(y-6)$, (2);
from which $x=78$, and $y=66$.

(33) Let $x=$ the length, and $y=$ the breadth.
Then by the 1st condition, $x+4 : y+4 :: 5 : 4$,
whence $4(x+4)=5(y+4)$, (1).
By the 2d condition, $x-4 : y-4 :: 4 : 3$,
whence $3(x-4)=4(y-4)$, (2);
from which $x=36$, and $y=28$.

(34) Let $x=$ number of acres of tillable, and $y=$ number of acres of pasture.
Then $200x+140y=24500$, (1),
Also, $x : x-y :: 14 : 9$,
whence $9x=14x-14y$, (2);
from which $x=98$, and $y=35$.

NOTE. In forming equation (1), it is important for the pupil to notice that the quantities on both sides must be expressed in the same denomination, which, in this case, is cents.

(35) Let $x=$ number of A's sheep, and $y=$ number of B's.
Then, by the 1st condition, $x+10 : y-20 :: 4 : 3$,
whence $3(x+10)=4(y-20)$, (1).
Again $x+10-20=x-10=$ number in A's flock at end of 2d year,
and $y-20+10=y-10=$ number in B's flock at end of 2d year.
Then $x-10 : y-10 :: 6 : 7$,
whence $7(x-10)=6(y-10)$, (2);
from which $x=70$, and $y=80$.

(36) Let $x=$ number of gallons in first, and $y=$ number in second.
By the 1st condition, $x-15=\frac{2}{3}(y-15)$, (1),
also, $x-15-25=x-40$, and $y-15-25=y-40$;
therefore, by the 2d condition, $x-40=\frac{1}{2}(y-40)$, (2);
from which $x=65$, and $y=90$.

(37) Let $x=$ the numerator, and $y=$ the denominator.
Then $\frac{x+1}{x+y}=\frac{1}{4}$, (1),
and $\frac{x+y}{y+1}=\frac{8}{7}$, (2);
from which $x=3$, and $y=13$.

(38) Representing the first two numbers by $5x$ and $7x$, and the other two by $3y$ and $5y$,
By the 1st condition we have $5x+3y : 7x+5y :: 9 : 13$,
whence $13(5x+3y)=9(7x+5y)$, (1);
the difference of their sums $=(7x+5y)-(5x+3y)=2x+2y$,
therefore $2x+2y=16$, (2);
from which $x=6$, and $y=2$;
hence $5x=30$, $7x=42$; $3y=6$, and $5y=10$.

(39) Let $x=$ number of bushels of rye, and $y=$ number of bushels of wheat.
Then $28+x+y=100$, (1).
And $28\times28+36x+48y=100\times40=4000$, (2);
from which $x=20$, and $y=52$.

(40) Since $4x$ and $5x$ have the same ratio as 4 and 5, let them represent the weights of the loaded wagons. Also, let $6y$ and $7y$, which have the same ratio as 6 and 7, represent the parts of the loads taken out.

Then $4x-6y : 5x-7y :: 2 : 3$,
whence $3(4x-6y)=2(5x-7y)$, (1),
and $4x-6y+5x-7y=10$, (2);
from which $x=4$, and $y=2$;
hence $4x=16$, and $5x=20$.

(41) Let $x=$ number of gallons in first, and $y=$ number of gallons in second.

First.	Second.
x	y
y	y
$x-y$	$2y=$ gals. in each after first pouring,
$x-y$	$x-y$
$2x-2y$	$3y-x=$ gals. in each after 2d pouring.
$3y-x$	$3y-x$
$3x-5y$	$6y-2x=$ gals. in each after 3d pouring.

Therefore, $3x-5y=6y-2x$, (1),
and $3x-5y=16$, (2);
from which $x=22$, and $y=10$.

This question may be easily solved by arithmetic, by reversing the operations, thus:

First.	Second.
16	16= gals. in each after 3d pouring;
8	8
24	8= gals. in each after 2d pouring;
12	12
12	20= gals. in each after 1st pouring;
10	10
22	10= gals. in each before 1st pouring.

It is evident that by the third pouring the number of gallons which the second vessel contained previously, was doubled; hence, by subtracting from it, half of what the second contained after the third pouring, and adding the same quantity to the first, we find what both contained previous to the second pouring. In the same manner, by subtracting from the first, half of what it contained after the second pouring, and adding the same quantity to the second, we find what each contained previous to the second pouring. Lastly, by subtracting from the second half of what it contained after the first pouring, and adding the same quantity to the first, we find what each originally contained.

QUESTIONS PRODUCING EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

Article 163.

(2) Let x, y, and z represent the numbers.

Then $x+y=27$, (1),

$x+z=32$, (2),

and $y+z=35$, (3);

from which $x=12$, $y=15$, and $z=20$.

(3) Let x, y, and z represent the numbers.

Then $x+y+z=59$, (1),

$\frac{x-y}{2}=5$, (2),

$\frac{x-z}{2}=9$, (3);

from which $x=29$, $y=19$, and $z=11$.

We have assumed that the first number is greater than the second or third; but a correct result will be as readily obtained by supposing the third number greater than the first or second.

(4) Let x, y, and z represent the numbers.

Then $x+\frac{1}{2}y=14$, (1),

$y+\frac{1}{3}z=18$, (2),

and $z+\frac{1}{4}x=20$, (3);

from which $x=8$, $y=12$, and $z=18$.

(5) Let x, y, and z represent the prices respectively of the three watches.

Then $x+\frac{y+z}{2}=25$, (1),

$y+\frac{x+z}{3}=26$, (2),

and $z+\frac{x+z}{2}=29$, (3);

from which $x=8$, $y=18$, and $z=16$.

(6) Let x, y, and z represent the three numbers.

Then $x+\frac{y+z}{3}=25$, (1),

$y+\frac{x+z}{4}=25$, (2),

and $z+\frac{x+y}{5}=25$, (3);

from which $x=13$, $y=17$, and $z=19$.

(7) Let $v=$ cost of an apple, $x=$ cost of a pear, $y=$ cost of a peach, and $z=$ cost of an orange.

Then $2v+5x=12$, (1),

$3x+4y=18$, (2),

$4x+5z=28$, (3),

and $5y+6z=39$, (4);

from which $v=1$, $x=2$, $y=3$, and $z=4$.

(8) Let $x=$ A's money, $y=$ B's, and $z=$ C's.

Then $x+y=\frac{2}{3}z$, (1),

$y+z=6x$, (2),

$x+z=y+680$, (3);

from which $x=200$, $y=360$, and $z=840$.

(9) Let $x=$ A's money, $y=$ B's, and $z=$ C's.

Then $x+y+z=1820$, (1),

$x+200=y-200+160$, (2),

and $y+70=z-70$, (3);

from which $x=400$, $y=640$, and $z=780$.

(10) Let $x=$ A's money, $y=$ B's, and $z=$ C's.

Then $x+700=2(y-700)$, (1),

$y+1400=3(z-1400)$, (2),

and $z+420=5(x-420)$, (3);

from which $x=980$, $y=1540$, and $z=2380$.

(11) Let x, y, and z, represent the digits in hundred's, ten's, and unit's places respectively.

Then $100x+10y+z$ represents the number.

Therefore $x+y+z=11$, (1),

$z=2x$, (2),

and $100x+10y+z+297=100z+10y+x$, (3);

from which $x=3$, $y=2$, and $z=6$.

(12) Let $x=$ A's money, $y=$ B's, and $z=$ C's.

Then $x+y+z=2000$, (1),

$y+200=z+100$, (2),

$y-100=\frac{3}{4}z$, (3),

from which $x=500$, $y=700$, and $z=800$.

(13) Let x, y, and z represent the numbers.
Then $x+y+z=83$, (1),
$x-7 : y-7 :: 5 : 3$
whence $3(x-7)=5(y-7)$, (2),
$y-3 : z-3 :: 11 : 9$
whence $9(y-3)=11(z-3)$, (3);
from which $x=37$, $y=25$, and $z=21$.

(14) Let $x=$ A's share, $y=$ B's, and $z=$ C's.
Then $x+y+z=180$, (1),
$2x+80=3y+40$, (2),
and $2x+80=4z+20$, (3);
from which $x=70$, $y=60$, and $z=50$.

(15) Let x, y, and z represent the numbers.
Then $x+y+z=78$, (1);
$\frac{1}{3}x : \frac{1}{4}y :: 1 : 2$,
whence $\frac{2}{3}x=\frac{1}{4}y$, (2);
$\frac{1}{4}y : \frac{1}{5}z :: 2 : 3$,
whence $\frac{3}{4}y=\frac{2}{5}z$, (3),
from which $x=9$, $y=24$, and $z=45$.

(16) Let $x=$ A's share, $y=$ B's, and $z=$ C's.
Then $x-\frac{4}{7}(y+z)=30$, (1),
$y-\frac{3}{8}(x+z)=30$, (2),
and $z-\frac{2}{9}(x+y)=30$, (3);
from which $x=150$, $y=120$, and $z=90$.

(17) Let x, y, and z, represent the days respectively, in which A, B and C each, alone, can perform the work.

Then since A can do it in x days, he can do $\frac{1}{x}$ part in 1 day. In like manner B can do $\frac{1}{y}$ part of it, and C, $\frac{1}{z}$ part of it in 1 day. Also, since A and B can perform the work in 12 days, they can perform $\frac{1}{12}$ of it in 1 one day. For a like reason A and C can perform $\frac{1}{15}$, and B and C $\frac{1}{20}$ in 1 day.

Therefore, $\frac{1}{x}+\frac{1}{y}=\frac{1}{12}$, (1),

$\frac{1}{x}+\frac{1}{z}=\frac{1}{15}$, (2),

and $\frac{1}{y}+\frac{1}{z}=\frac{1}{20}$, (3).

By subtracting Eq. (2) from (1) we have

$$\frac{1}{y}-\frac{1}{z}=\tfrac{1}{12}-\tfrac{1}{15}=\tfrac{1}{60}, \qquad (4);$$

by adding equations (4) and (3) together

$$\frac{2}{y}=\tfrac{1}{20}+\tfrac{1}{60}=\tfrac{1}{15}, \qquad (5),$$

or $y=30$, by clearing of fractions.

By subtracting equation (4) from (3)

$$\frac{2}{z}=\tfrac{1}{20}-\tfrac{1}{60}=\tfrac{1}{30}, \qquad (6),$$

or $z=60$, by clearing of fractions.

The value of x may be found by substituting the value of y in Eq. (1), or by subtracting Eq. (3) from (2), and adding the resulting equation and Eq. (1) together.

NOTE. For another method of finding the values of the unknown quantities after the equations are formed, see Art. 169, Example 1st.

(18) Let $x=$ digit in hundred's place, $y=$ digit in ten's place, and $z=$ digit in unit's place.

Then $100x+10y+z=$ the number,

$$\text{also } \frac{100x+10y+z}{x+y+z+9}=19, \quad (1),$$

$$y=\frac{x+z}{2}, \quad (2),$$

and $100x+10y+z+198=100z+10y+x$, (3), from which $x=4$, $y=5$, and $z=6$.

(19) Let $x=$ bushels of barley, $y=$ bushels of rye, and $z=$ bushels of wheat.

Then $x+y+z=100$, (1),

$28x+36y+48z=4000$, (2),

$28x+36\times2y+48(z+10)=40(100+y+10)$,

by reducing, $28x+32y+48z=3920$, (3);

from which $x=28$, $y=20$, and $z=52$.

(20) Let x, y, and z represent the birds respectively which A, B, and C killed.

x	y	z	
$y+z$	y	z	
$x-y-z$	$2y$	$2z$,	after 1st division;
$x-y-z$	$x-y-z+2z$	$2z$	
$2x-2y-2z$	$3y-x-z$	$4z$,	after 2d division;
$2x-2y-2z$	$3y-x-z$	$x+y-3z$	
$4x-4y-4z$,	$6y-2x-2z$,	$7z-x-y$,	after 3d division.

Therefore $x+y+z=96$, (1),
$4x-4y-4z=32$, (2),
$6y-2x-2z=32$, (3);
from which $x=52$, $y=28$, and $z=16$

By reversing the operation, as in the solution of example 41 page 31, this question is easily solved by arithmetic thus:

A	B	C.	
32	32	32,	at close of 3d division;
16	16	32	
16	16	64	at close of 2d division;
8	40	32	
8	56	32	at close of 1st division;
44	28	16	
52	28	16	previous to the 1st division.

Since each had an equal number after the third division, therefore, each must have had $\frac{1}{3}$ of 96, which is 32. And since, in making the third division C gave to A and B as many as they had, it is evident that *before* the third division, that is, *after* the second division, A and B must each have had $\frac{1}{2}$ of 32, which is 16, and C 32, and what A and B received at the third division, making in all 64. By reasoning in a similar manner we find what each had previous to the other divisions.

GENERALIZATION.

Article 170.

GENERAL PROBLEMS.

(1) Let $x=$ one of the parts, then $a-x$ will be the other.
Therefore, $x=n(a-x)=na-nx$,
transposing, $nx+x=na$,
factoring, $(n+1)x=na$,
dividing, $x=\frac{na}{n+1}$;

$$a-x=a-\frac{na}{n+1}=\frac{na+a-na}{n+1}=\frac{a}{n+1}.$$

(2) Let $x=$ one of the parts, then $a-x$ will be the other.
Therefore, $mx=n(a-x)=na-nx$,
transposing, $mx+nx=na$,

factoring, $(m+n)x=na$,

dividing, $x=\frac{na}{m+n}$;

$$a-x=\frac{ma+na}{m+n}-\frac{na}{m+n}=\frac{ma}{m+n}.$$

(3) Let $x=$ one part, then $a-x$ will be the other.

Then $mx+n(a-x)=b$.

trans. and fact'ng, $(m-n)x=b-na$,

dividing, $x=\frac{b-na}{m-n}$;

$$a-x=\frac{ma-na}{m-n}-\frac{b-na}{m-n}=\frac{ma-b}{m-n}$$

(4) Let $x=$ the number.

Then $\frac{x}{m}+\frac{x}{n}=a$,

$nx+mx=mna$, by clearing of fractions,

whence $x=\frac{mna}{m+n}$.

(5) Let $x=$ the first part; then $mx=$ the second, and $nx=$ the third part.

Therefore, $x+mx+nx=a$,

factoring, $(1+m+n)x=a$,

$$x=\frac{a}{1+m+n};$$

$$mx=\frac{ma}{1+m+n};$$

$$nx=\frac{na}{1+m+n}.$$

(6) Let $x=$ one part, then $a-x=$ the other.

Then $\frac{x}{b}+\frac{a-x}{c}=d$;

$cx+ab-bx=bcd$, by clearing of fractions,

$cx-bx=bcd-ab$, by transposing,

or, $bx-cx=ab-bcd$, by changing the signs of all the terms on both sides.

$(b-c)x=b(a-cd)$,

$$x=\frac{b(a-cd)}{b-c};$$

$$a-x=\frac{a(b-c)}{b-c}-\frac{b(a-cd)}{b-c}=\frac{c(bd-a)}{b-c}.$$

(7) Let $x=$ the number.
Then $a+x : b+x :: m : n$,
whence $n(a+x)=m(b+x)$;
transposing, $nx-mx=mb-na$,

$$x=\frac{mb-na}{n-m}.$$

(8) Let $x=$ the number.
Then $a-x : b-x :: m : n$,
whence $n(a-x)=m(b-x)$.
transposing, $mx-nx=mb-na$,

$$x=\frac{mb-na}{m-n}, \text{ or } \frac{na-mb}{n-m}.$$ See Art. 132

(9) Let $x=$ the number.
Then $a+x \quad b-x :: m : n$,
whence $n(a+x)=m(b-x)$,
transposing, $mx+nx=mb-na$,

$$x=\frac{mb-na}{m+n}.$$

(10) Let $x=$ the number of dollars he had at first.

Then $x-\frac{1}{m}x-\frac{1}{n}x=a$,

$mnx-nx-mx=mna$, by clearing of fractions;
$(mn-m-n)x=mna$,

$$x=\frac{mna}{mn-m-n}.$$

(11) Let $x=$ the quantity.

Then $\frac{m}{n}x-\frac{p}{q}x=a$,

$mqx-npx=anq$, by clearing of fractions;

$$x=\frac{anq}{mq-np}.$$

(12) Let $x=$ the number of persons.
Then $ax=$ the number of cents paid;
also $(x-b)c=$ the number of cents paid;
therefore, $(x-b)c=ax$;
$cx-ax=bc$,

$$x=\frac{bc}{c-a}.$$

(13) Let $x=$ the number of persons.
Then $ax+b=$ the number of cents the person had,
also $cx+d=$ the number of cents the person had;
Therefore $ax+b=cx+d$;

$$(a-c)x=d-b, \text{ by transposing;}$$

$$x=\frac{d-b}{a-c}.$$

(14) Let $x=$ the number of bushels of oats, then $n-x=$ the number of bushels of rye.
Then $ax=$ cost of x bushels at a cents per bushel;
$(n-x)b=$ cost of $n-x$ bushels at b cents per bushel;
therefore $ax+(n-x)b=nc$,

$$(a-b)x=nc-nb=n(c-b),$$

$$x=\frac{n(c-b)}{a-b};$$

$$n-x=\frac{n(a-b)}{a-b}-\frac{n(c-b)}{a-b}=\frac{n(a-c)}{a-b}.$$

(15) Let $x=$ the money he had in his purse.
$x+x=2x$, then $2x-a=$ money he had after 1st spending;
$2x-a+(2x-a)-a=4x-3a=$ money after 2d spnd'g;
$4x-3a+(4x-3a)-a=8x-7a=$ money after 3d spnd'g;
$8x-7a+(8x-7a)-a=16x-15a=$ money after 4th "
Therefore $16x-15a=0$,

$$16x=15a,$$

$$x=\tfrac{15}{16}a.$$

(16) Let $x=$ number of pieces of 1st kind, then $c-x=$ number of pieces of second kind.
Since a pieces of the first kind make 1 dollar, or 100 cts.
therefore $\frac{100}{a}=$ value in cents of a piece of the first kind.
In like manner $\frac{100}{b}=$ value in cents of a piece of the second.
$\frac{100}{a}x=$ value in cents of x pieces of first kind.
$\frac{100}{b}(c-x)=$ value in cents of $(c-x)$ pieces of the second kind.

Therefore $\frac{100}{a}x+\frac{100}{b}(c-x)=100$;

$\frac{x}{a}+\frac{c-x}{b}=1$, by dividing both sides by 100;

$bx+ac-ax=ab$, by clearing of fractions.

$$(b-a)x=a(b-c);$$

$$x=\frac{a(b-c)}{b-a}.$$

$$c-x=\frac{c(b-a)}{b-a}-\frac{a(b-c)}{b-a}=\frac{b(c-a)}{b-a}.$$

To illustrate this question by numbers, take the following: How many 5 and 25 cent pieces must be taken, so that 8 shall make a dollar?

Ans. 5 five cent pieces, and 3 twenty-five cent pieces.

Article 171.

(11) Let $x=$ the less number; then $x+2=$ the greater. Therefore $x(x+2)=x^2+8$,

whence $x=4$, and $x+2=6$.

(12) Let $x=$ the greater part, then $a-x=$ the less.

Therefore $x^2-(a-x)^2=c$

whence $x=\frac{a^2+c}{2a}$, and $a-x=\frac{a^2-c}{2a}$.

(13) Let $x=$ number of pages, and $y=$ number of lines on a page; then $xy=$ number of lines in the book.

Therefore $(x+5)(y+10)=xy+450$, (1),

and $(x-10)(y-5)=xy-450$, (2);

from which we find $x=20$, and $y=40$.

Article 172.

NEGATIVE SOLUTIONS.

Enunciation of questions 2, 3, 4, and 5, so that the results will be true in an arithmetical sense.

2. What number must be *added* to 20, that the *sum* may be 25?

Ans. 5.

3. What number must be *subtracted* from 11, that the *remainder* being multiplied by 5, the product shall be 40?

Ans. 3

4. What number is that, of which the $\frac{2}{3}$ is *less* than the the $\frac{3}{4}$ by 3? *Ans.* 36.

5 A father, whose age is 45 years, has a son aged 15; *how many years since*, was the son $\frac{1}{4}$ as old as his father?

Ans. 5.

RADICALS OF THE SECOND DEGREE.

NOTE. All the examples in the FORMATION OF POWERS, and EXTRACTION OF THE SQUARE ROOT, being performed by direct, straightforward methods of operation, can present but few difficulties, if any, to the careful student. In the examples Art. 196, before commencing the operation the pupil must be careful to arrange the terms of the polynomial with reference to a certain letter.

Article 199.

REDUCTION OF RADICALS OF THE SECOND DEGREE.

(1) $\sqrt{8a^2}=\sqrt{4a^2\times 2}=\sqrt{4a^2}\times\sqrt{2}=2a\sqrt{2}.$

(2) $\sqrt{12a^3}=\sqrt{4a^2\times 3a}=\sqrt{4a^2}\times\sqrt{3a}=2a\sqrt{3a}.$

(3) $\sqrt{16a^3b}=\sqrt{16a^2\times ab}=\sqrt{16a^2}\times\sqrt{ab}=4a\sqrt{ab}.$

(4) $\sqrt{18a^4b^3c^3}=\sqrt{9a^4b^2c^2\times 2bc}=\sqrt{9a^4b^2c^2}\times\sqrt{2bc}$
$=3a^2bc\sqrt{2bc}.$

(5) $\sqrt{20a^3b^3c^3}=\sqrt{4a^2b^2c^2\times 5abc}=\sqrt{4a^2b^2c^2}\times\sqrt{5abc}$
$=2abc\sqrt{5abc}.$

REMARK. It is not necessary that the second step of the operation should always be written down, as in the preceding solutions; it should be done, however, by the pupil, on the slate or blackboard, until the principles are well understood

(6) $3\sqrt{24a^4c^2}=3\sqrt{4a^4c^2\times 6}=3\times 2a^2c\sqrt{6}=6a^2c\sqrt{6}.$

(7) $4\sqrt{27a^3c^3}=4\sqrt{9a^2c^2\times3ac}=4\times3ac\sqrt{3ac}=12ac\sqrt{3ac}.$

(8) $7\sqrt{28a^5c^2}=7\sqrt{4a^4c^2\times7a}=7\times2a^2c\sqrt{7a}=14a^2c\sqrt{7a}.$

(9) $\sqrt{32a^6b^2c^4}=\sqrt{16a^6b^2c^4\times2}=$ - - $4a^3bc^2\sqrt{2}.$

(10) $\sqrt{40a^2b^3c^5}=\sqrt{4a^2b^2c^4\times10bc}=$ - - $2abc^2\sqrt{10bc}.$

(11) $\sqrt{44a^5b^3c}=\sqrt{4a^4b^2\times11abc}=$ - - $2a^2b\sqrt{11abc}$

(12) $\sqrt{45a^4b^6c^4}=\sqrt{9a^4b^6c^4\times5}=$ - - $3a^2b^3c^2\sqrt{5}.$

(13) $\sqrt{48a^8b^6c^4}=\sqrt{16a^8b^6c^4\times3}=$ - - $4a^4b^3c^2\sqrt{3}.$

(14) $\sqrt{75a^3b^3c^3}=\sqrt{25a^2b^2c^2\times3abc}=$ - - $5abc\sqrt{3abc}.$

(15) $\sqrt{128a^6b^4c^2}=\sqrt{64a^6b^4c^2\times2}=$ - - $8a^3b^2c\sqrt{2}.$

(16) $\sqrt{243a^3b^2c}=\sqrt{81a^2b^2\times3ac}=$ - $9ab\sqrt{3ac}.$

(18) $\sqrt{\frac{3}{5}}=\sqrt{\frac{3}{5}\times\frac{5}{5}}=\sqrt{\frac{1}{25}\times15}=$ - - $\frac{1}{5}\sqrt{15}.$

(19) $\sqrt{\frac{7}{8}}=\sqrt{\frac{7}{8}\times\frac{2}{2}}=\sqrt{\frac{1}{16}\times14}=$ - - $\frac{1}{4}\sqrt{14}.$

(20) $\sqrt{\frac{12}{25}}=\sqrt{\frac{4}{25}\times3}=$ - - - - $\frac{2}{5}\sqrt{3}.$

(21) $\sqrt{\frac{11}{18}}=\sqrt{\frac{11}{18}\times\frac{2}{2}}=\sqrt{\frac{1}{36}\times22}=$ - - $\frac{1}{6}\sqrt{22}.$

(22) $9\sqrt{\frac{16}{27}}=9\sqrt{\frac{16}{27}\times\frac{3}{3}}=9\sqrt{\frac{16}{81}\times3}=9\times\frac{4}{9}\sqrt{3}=4\sqrt{3}.$

(23) $5\sqrt{\frac{9}{10}}=5\sqrt{\frac{9}{10}\times\frac{10}{10}}=5\sqrt{\frac{9}{100}\times10}=5\times\frac{3}{10}\sqrt{10}=\frac{3}{2}\sqrt{10}.$

(24) $10\sqrt{\frac{3}{50}}=10\sqrt{\frac{3}{50}\times\frac{2}{2}}=10\sqrt{\frac{1}{100}\times6}=10\times\frac{1}{10}\sqrt{6}=\sqrt{6}.$

(25) $7\sqrt{\frac{3}{28}}=7\sqrt{\frac{3}{28}\times\frac{7}{7}}=7\sqrt{\frac{1}{196}\times21}=7\times\frac{1}{14}\sqrt{21}=\frac{1}{2}\sqrt{21}.$

(26) $5=\sqrt{5\times5}=$ - - - - - $\sqrt{25}.$

(27) $2a=\sqrt{2a\times2a}=$ - - - - - $\sqrt{4a^2}.$

(28) $3\sqrt{5}=\sqrt{3\times3\times\sqrt{5}}=\sqrt{3\times3\times5}=$ - - $\sqrt{45}.$

(29) $3c\sqrt{2c}=\sqrt{3c\times3c\times2c}=$ - - - - $\sqrt{18c^3}$

(30) $5\sqrt{3}=\sqrt{5\times5\times3}=$ - - - $\sqrt{75}.$

Article 200.

ADDITION OF RADICALS OF THE SECOND DEGREE.

(3) $\sqrt{8}=2\sqrt{2}$
$\sqrt{18}=3\sqrt{2}$
Sum $=5\sqrt{2}$

(4) $\sqrt{12}=2\sqrt{3}$
$\sqrt{27}=3\sqrt{3}$
Sum $=5\sqrt{3}$

(5) $\sqrt{20}=2\sqrt{5}$
$\sqrt{80}=4\sqrt{5}$
Sum $=6\sqrt{5}$

(6) $\sqrt{24}=2\sqrt{6}$
$\sqrt{150}=5\sqrt{6}$
Sum $=7\sqrt{6}$

(7) $\sqrt{8}=2\sqrt{2}$
$\sqrt{32}=4\sqrt{2}$
$\sqrt{50}=5\sqrt{2}$
Sum $=11\sqrt{2}$

(8) $\sqrt{40}=2\sqrt{10}$
$\sqrt{90}=3\sqrt{10}$
$\sqrt{250}=5\sqrt{10}$
Sum $=10\sqrt{10}$

(9) $\sqrt{28a^2b^2}=2ab\sqrt{7}$
$\sqrt{112a^2b^2}=4ab\sqrt{7}$
Sum $=6ab\sqrt{7}$

(10) $\sqrt{75a^2c}=5a\sqrt{3c}$
$\sqrt{147a^2c}=7a\sqrt{3c}$
Sum $=12a\sqrt{3c}$

(11) $\sqrt{\frac{1}{3}}=\frac{1}{3}\sqrt{3}$
$\sqrt{\frac{3}{25}}=\frac{1}{5}\sqrt{3}$
Sum $=\frac{8}{15}\sqrt{3}$

(12) $\sqrt{\frac{1}{5}}=\frac{1}{5}\sqrt{5}$
$\sqrt{\frac{5}{49}}=\frac{1}{7}\sqrt{5}$
Sum $=\frac{12}{35}\sqrt{5}$

(13) $\sqrt{\frac{1}{2}}=\frac{1}{2}\sqrt{2}$
$\sqrt{8}=2\sqrt{2}$
Sum $=2\frac{1}{2}\sqrt{2}$

(14) $2\sqrt{\frac{3}{4}}=\sqrt{3}$
$3\sqrt{12}=6\sqrt{3}$
Sum $=7\sqrt{3}$

(15) $\frac{1}{2}\sqrt{\frac{1}{2}}=\frac{1}{4}\sqrt{2}$
$\frac{3}{4}\sqrt{2}=\frac{3}{4}\sqrt{2}$
Sum $=\sqrt{2}$

(16) $3\sqrt{\frac{2}{3}}=\sqrt{6}$
$7\sqrt{\frac{27}{50}}=\frac{21}{10}\sqrt{6}$
Sum $=\frac{31}{10}\sqrt{6}$

(17) $\sqrt{48a^2c^2x}=4ac\sqrt{3x}$
$\sqrt{12b^2x}=2b\sqrt{3c}$
Sum $=(4ac+2b)\sqrt{3c}$

(18) $\sqrt{(2a^3-4a^2c+2ac^2)}=\sqrt{(a^2-2ac+c^2)\times 2a}=(a-c)\sqrt{2a}$
$\sqrt{(2a^3+4a^2c+2ac^2)}=\sqrt{(a^2+2ac+c^2)\times 2a}=(a+c)\sqrt{2a}$
Sum $=2a\sqrt{2a}$

(19) $\sqrt{a+x}=\sqrt{a+x}$
$\sqrt{ax^2+x^3}=x\sqrt{a+x}$
$\sqrt{(a+x)^3}=(a+x)\sqrt{a+x}$
Sum $=(1+a+2x)\sqrt{a+x}$

Article 201.

SUBTRACTION OF RADICALS OF THE SECOND DEGREE.

(2) $\sqrt{18}=3\sqrt{2}$
$\sqrt{2}=\sqrt{2}$
Dif. $=2\sqrt{2}$

(3) $\sqrt{45a^2}=3a\sqrt{5}$
$\sqrt{5a^2}=a\sqrt{5}$
Dif. $=2a\sqrt{5}$

(4) $\sqrt{54b}=3\sqrt{6b}$
$\sqrt{6b}=\sqrt{6b}$
Dif. $=2\sqrt{6b}$

(5) $\sqrt{112a^2c^2}=4ac\sqrt{7}$
$\sqrt{28a^2c^2}=2ac\sqrt{7}$
Dif. $=2ac\sqrt{7}$

(6) $\sqrt{27b^3c^3}=3bc\sqrt{3bc}$
$\sqrt{12b^3c^3}=2bc\sqrt{3bc}$
Dif. $=bc\sqrt{3bc}$

(7) $\sqrt{36a^5}=6a^2\sqrt{a}$
$\sqrt{4a^5}=2a^2\sqrt{a}$
Dif. $=4a^2\sqrt{a}$

(8) $\sqrt{49ab^3c^2}=7bc\sqrt{ab}$
$\sqrt{25ab^3c^2}=5bc\sqrt{ab}$
Dif. $=2bc\sqrt{ab}$

(9) $\sqrt{160a^3b^3c}=4ab\sqrt{10abc}$
$\sqrt{10a^3b^3c}=ab\sqrt{10abc}$
Dif. $=3ab\sqrt{10abc}$

(10) $5a\sqrt{27}=15a\sqrt{3}$
$3a\sqrt{48}=12a\sqrt{3}$
Dif. $=3a\sqrt{3}$

(11) $2\sqrt{\frac{3}{4}}=\sqrt{3}$
$3\sqrt{\frac{1}{3}}=\sqrt{3}$
Dif. $=0$

(12) $\sqrt{\frac{5}{6}}=\frac{1}{6}\sqrt{30}$
$\sqrt{\frac{10}{27}}=\frac{1}{9}\sqrt{30}$
Dif. $=\frac{1}{18}\sqrt{30}$

(13) $\sqrt{12}=2\sqrt{3}$
$\sqrt{\frac{3}{4}}=\frac{1}{2}\sqrt{3}$
Dif. $=\frac{3}{2}\sqrt{3}$

(14) $3\sqrt{\frac{1}{2}}=\frac{3}{2}\sqrt{2}$
$\sqrt{2}=\sqrt{2}$
Dif. $=\frac{1}{2}\sqrt{2}$

(15) $\sqrt{\frac{2}{3}}=\frac{1}{3}\sqrt{6}$
$\sqrt{\frac{2}{27}}=\frac{1}{9}\sqrt{6}$
Dif. $=\frac{2}{9}\sqrt{6}$

(16) $\sqrt{4a^2x}=2a\sqrt{x}$
$a\sqrt{x^3}=ax\sqrt{x}$
Dif. $=(2a-ax)\sqrt{x}$

(17) $\sqrt{3m^2x+6mnx+3n^2x}=(m+n)\sqrt{3x}$
$\sqrt{3m^2x-6mnx+3n^2x}=(m-n)\sqrt{3x}$
Dif. $=2n\sqrt{3x}$

Article 202.

MULTIPLICATION OF RADICALS OF THE SECOND DEGREE.

(3) $\sqrt{8}\times\sqrt{2}=\sqrt{16}=4.$

(4) $2\sqrt{a}\times3\sqrt{a}=6\sqrt{a^2}=6a.$

(5) $\sqrt{27}\times\sqrt{3}=\sqrt{81}=9.$

(6) $3\sqrt{2}\times2\sqrt{3}=6\sqrt{6}.$

(7) $3\sqrt{3}\times2\sqrt{3}=6\sqrt{9}=18.$

(8) $\sqrt{6}\times\sqrt{15}=\sqrt{90}=3\sqrt{10}$

(9) $2\sqrt{15}\times3\sqrt{35}=6\sqrt{3\times5\times5\times7}=30\sqrt{21}.$

(10) $\sqrt{a^3b^5c}\times\sqrt{abc}=\sqrt{a^4b^6c^2}=a^2b^3c.$

(11) $\sqrt{\frac{1}{3}}\times\sqrt{\frac{3}{4}}=\sqrt{\frac{1}{4}}=\frac{1}{2}.$

(12) $\sqrt{\frac{2}{5}}\times\sqrt{\frac{8}{9}}=\sqrt{\frac{16}{45}}=\sqrt{\frac{16}{225}\times 5}=\frac{4}{15}\sqrt{5}.$

(13) $2\sqrt{\frac{a}{5}}\times 3\sqrt{\frac{a}{10}}=6\sqrt{\frac{a^2}{50}}=6\sqrt{\frac{a^2}{100}\times 2}=\frac{6a}{10}\sqrt{2}=\frac{3a}{5}\sqrt{2}.$

(14)
$$\begin{array}{r} 2+\sqrt{2} \\ 2-\sqrt{2} \\ \hline 4+2\sqrt{2} \\ -2\sqrt{2}-2 \\ \hline 4-2=2. \end{array}$$

(15)
$$\begin{array}{r} 1+\sqrt{2} \\ 1-\sqrt{2} \\ \hline 1+\sqrt{2} \\ -\sqrt{2}-2 \\ \hline 1-2=-1. \end{array}$$

(16) $\sqrt{x+2}\times\sqrt{x-2}=\sqrt{(x+2)(x-2)}=\sqrt{x^2-4}.$

(17) $\sqrt{a+x}\times\sqrt{a+x}=\sqrt{(a+x)(a+x)}=a+x.$

(18) $\sqrt{ab+bx}\times\sqrt{ab-bx}=\sqrt{(ab+bx)(ab-bx)}=\sqrt{a^2b^2-b^2x^2}$

(19) $\sqrt{x+2}\times\sqrt{x+3}=\sqrt{(x+2)(x+3)}=\sqrt{x^2+5x+6}.$

(20)
$$\begin{array}{l} c\sqrt{a}+d\sqrt{b} \\ c\sqrt{a}-d\sqrt{b} \\ \hline c^2a+cd\sqrt{ab} \\ \quad -cd\sqrt{ab}-d^2b \\ \hline c^2a \qquad\qquad -d^2b. \end{array}$$

(21)
$$\begin{array}{l} 7+2\sqrt{6} \\ 9-5\sqrt{6} \\ \hline 63+18\sqrt{6} \\ \quad -35\sqrt{6}-10\times 6 \\ \hline 3-17\sqrt{6}. \end{array}$$

(22)
$$\begin{array}{l} \sqrt{a+x}+\sqrt{a-x} \\ \sqrt{a+x}-\sqrt{a-x} \\ \hline (a+x)+\sqrt{(a^2-x^2)} \\ \qquad -\sqrt{(a^2-x^2)}-(a-x) \\ \hline (a+x)-(a-x)=2x. \end{array}$$

(23)
$$\begin{array}{l} x+2\sqrt{ax}+a \\ x-2\sqrt{ax}+a \\ \hline x^2+2x\sqrt{ax}+ax \\ \quad -2x\sqrt{ax}-4ax-2a\sqrt{ax} \\ \qquad\qquad +ax+2a\sqrt{ax}+a^2 \\ \hline x^2 \qquad\qquad -2ax \qquad\qquad +a^2 \end{array}$$

(24)
$$\begin{array}{l} x^2-x\sqrt{2}+1 \\ x^2+x\sqrt{2}+1 \\ \hline x^4-x^3\sqrt{2}+x^2 \\ \quad +x^3\sqrt{2}-2x^2+x\sqrt{2} \\ \qquad\qquad +x^2-x\sqrt{2}+1 \\ \hline x^4 \qquad\qquad\qquad\qquad +1. \end{array}$$

Article 203.

DIVISION OF RADICALS OF THE SECOND DEGREE.

(2) $\frac{\sqrt{54}}{\sqrt{6}}=\sqrt{\frac{54}{6}}=\sqrt{9}=3.$ (3) $\frac{6\sqrt{54}}{3\sqrt{27}}=\frac{6}{3}\sqrt{\frac{54}{27}}=2\sqrt{2}.$

(4) $\frac{6\sqrt{28}}{2\sqrt{7}}=\frac{6}{2}\sqrt{\frac{28}{7}}=3\sqrt{4}=6.$

(5) $\frac{\sqrt{160}}{\sqrt{8}}=\sqrt{\frac{160}{8}}=\sqrt{20}=2\sqrt{5}.$

(6) $\frac{15\sqrt{378}}{5\sqrt{6}}=\frac{15}{5}\sqrt{\frac{378}{6}}=3\sqrt{63}=3\sqrt{9\times 7}=9\sqrt{7}.$

(7) $\frac{\sqrt{a^3}}{\sqrt{a}}=\sqrt{\frac{a^3}{a}}=\sqrt{a^2}=a.$

(8) $\frac{ab\sqrt{a^3b^3}}{b\sqrt{ab}}=\frac{ab}{b}\sqrt{\frac{a^3b^3}{ab}}=a\sqrt{a^2b^2}=a^2b.$

(9) $\frac{a}{\sqrt{a}}=\frac{\sqrt{a^2}}{\sqrt{a}}=\sqrt{\frac{a^2}{a}}=\sqrt{a}.$

(10) $\frac{a\sqrt{b}}{c\sqrt{d}}=\frac{a}{c}\sqrt{\frac{b}{d}}=\frac{a}{c}\sqrt{\frac{bd}{d^2}}=\frac{a}{c}\sqrt{\frac{1}{d^2}\times bd}=\frac{a}{cd}\sqrt{bd}.$

(11) $\sqrt{\frac{a}{b}}\div\sqrt{\frac{d}{c}}=\sqrt{\frac{a}{b}}\times\frac{c}{d}=\sqrt{\frac{ac}{bd}}=\sqrt{\frac{acbd}{b^2d^2}}=\frac{1}{bd}\sqrt{abcd}.$

(12) $\sqrt{\frac{1}{2}}\div\sqrt{\frac{1}{3}}=\sqrt{\frac{1}{2}\times\frac{3}{1}}=\sqrt{\frac{3}{2}}=\sqrt{\frac{6}{4}}=\sqrt{\frac{1}{4}\times 6}=\frac{1}{2}\sqrt{6}.$

(13) $\sqrt{\frac{3}{4}}\div\sqrt{\frac{1}{3}}=\sqrt{\frac{3}{4}\times\frac{3}{1}}=\sqrt{\frac{9}{4}}=\frac{3}{2}=1\frac{1}{2}.$

(14) $\frac{2}{3}\sqrt{18}\div\frac{1}{2}\sqrt{2}=\frac{2}{3}\times\frac{2}{1}\sqrt{\frac{18}{2}}=\frac{4}{3}\sqrt{9}=4.$

(15) $\frac{3}{5}\sqrt{\frac{1}{3}}\div\frac{1}{2}\sqrt{\frac{3}{5}}=\frac{3}{5}\times\frac{2}{1}\sqrt{\frac{1}{3}\times\frac{5}{3}}=\frac{6}{5}\sqrt{\frac{1}{9}\times 5}=\frac{6}{15}\sqrt{5}=\frac{2}{5}\sqrt{5}.$

(16) $\frac{1}{2}\sqrt{\frac{1}{2}}=\frac{1}{2}\sqrt{\frac{1}{4}\times 2}=\frac{1}{2}\times\frac{1}{2}\sqrt{2}=\frac{1}{4}\sqrt{2}$;

$\sqrt{2}+3\sqrt{\frac{1}{2}}=\sqrt{2}+3\sqrt{\frac{1}{4}\times 2}=\sqrt{2}+\frac{3}{2}\sqrt{2}=\frac{5}{2}\sqrt{2}$

$\frac{1}{4}\sqrt{2}\div\frac{5}{2}\sqrt{2}=\frac{1}{4}\times\frac{2}{5}\sqrt{\frac{2}{2}}=\frac{2}{20}=\frac{1}{10}$.

Article 204.

(1) $\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{2}=\frac{1}{2}\sqrt{2}$.

(2) $\frac{\sqrt{2}}{\sqrt{3}}=\frac{\sqrt{2}}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{6}}{3}=\frac{1}{3}\sqrt{6}$.

(3) $\frac{1}{2+\sqrt{3}}=\frac{1}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}=\frac{2-\sqrt{3}}{4-3}=2-\sqrt{3}$.

(4) $\frac{3}{6-\sqrt{3}}=\frac{3}{6-\sqrt{3}}\times\frac{6+\sqrt{3}}{6+\sqrt{3}}=\frac{3(6+\sqrt{3})}{36-3}=\frac{1}{11}(6+\sqrt{3})$.

(5) $\frac{5}{\sqrt{7}+\sqrt{6}}=\frac{5}{\sqrt{7}+\sqrt{6}}\times\frac{\sqrt{7}-\sqrt{6}}{\sqrt{7}-\sqrt{6}}=\frac{5(\sqrt{7}-\sqrt{6})}{7-6}$.
$=5(\sqrt{7}-\sqrt{6})$.

(6) $\frac{8}{\sqrt{5}-\sqrt{3}}=\frac{8}{\sqrt{5}-\sqrt{3}}\times\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}=\frac{8(\sqrt{5}+\sqrt{3})}{5-3}$.
$=4(\sqrt{5}+\sqrt{3})$.

(7) $\frac{3}{\sqrt{5}}=\frac{3}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}=\frac{3}{5}\sqrt{5}=\frac{3}{5}(2.2360679+)=1.3416407+$.

(8) $\frac{3}{\sqrt{5}-\sqrt{2}}=\frac{3}{\sqrt{5}-\sqrt{2}}\times\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}=\frac{3(\sqrt{5}+\sqrt{2})}{3}$.
$=\sqrt{5}+\sqrt{2}=2.2360679+1.4142135+=3.650281+$.

(9) $\frac{\sqrt{2}}{\sqrt{5}-\sqrt{3}}=\frac{\sqrt{2}}{\sqrt{5}-\sqrt{3}}\times\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}=\frac{\sqrt{10}+\sqrt{6}}{5-3}$.
$=\frac{1}{2}(\sqrt{10}+\sqrt{6})=\frac{1}{2}(3.162277+2.449489+)=2.805833+$

Article 205.

SIMPLE EQUATIONS CONTAINING RADICALS OF THE SECOND DEGREE.

(1) $\sqrt{(x+3)}+3=7.$

Transposing, $\sqrt{(x+3)}=4;$
squaring, $x+3=16;$
from which $x=13.$

(2) $x+\sqrt{(x^2+11)}=11.$

Transposing, $\sqrt{(x^2+11)}=11-x;$
squaring, $x^2+11=121-22x+x^2;$
from which $x=5.$

(3) $\sqrt{(6+\sqrt{x-1})}=3.$

Squaring, $6+\sqrt{x-1}=9;$
transposing, $\sqrt{x-1}=3;$
squaring, $x-1=9;$
from which $x=10.$

(4) $\sqrt{x(a+x)}=a-x.$

Squaring, $x(a+x)=a^2-2ax+x^2.$
reducing $3ax=a^2;$
from which $x=\frac{a}{3}.$

(5) $\sqrt{x}-2=\sqrt{(x-8)}.$

Squaring, $x-4\sqrt{x}+4=x-8;$
reducing, $-4\sqrt{x}=-12;$
dividing, $\sqrt{x}=3;$
squaring, $x=9.$

(6) $x+\sqrt{x^2-7}=7.$

Transposing, $\sqrt{x^2-7}=7-x;$
squaring, $x^2-7=49-14x+x^2;$
from which $x=4.$

(7) $2+\sqrt{3x}=\sqrt{5x+4}.$

Squaring, $4+4\sqrt{3x}+3x=5x+4;$
reducing, $4\sqrt{3x}=2x;$
dividing, $2\sqrt{3x}=x:$
squaring, $4\times3x=x^2;$
from which $x=12.$

(8) $\sqrt{x+7}=6-\sqrt{x-5}$.

Squaring, $x+7=36-12\sqrt{x-5}+x-5$,

transposing and reducing, $\sqrt{x-5}=2$;

squaring, $x-5=4$;

from which $x=9$.

(9) $\sqrt{x-a}=\sqrt{x}-\frac{1}{2}\sqrt{a}$.

Squaring, $x-a=x-\sqrt{ax}+\frac{1}{4}a$;

transposing, and reducing, $\sqrt{ax}=\frac{5a}{4}$;

squaring, $ax=\frac{25a^2}{16}$;

whence $x=\frac{25a}{16}$.

(10) $\sqrt{x+225}-\sqrt{x-424}-11=0$.

Transposing, $\sqrt{x+225}=11+\sqrt{x-424}$;

squaring, $x+225=121+22\sqrt{x-424}+x-424$;

reducing, $528=22\sqrt{x-424}$;

dividing, $24=\sqrt{x-424}$;

squaring, $576=x-424$;

from which $x=1000$.

(11) $x+\sqrt{2ax+x^2}=a$.

Transposing, $\sqrt{2ax+x^2}=a-x$;

squaring, $2ax+x^2=a^2-2ax+x^2$;

reducing, $4ax=a^2$;

whence $x=\frac{1}{4}a$.

(12) $\sqrt{x+a}-\sqrt{x-a}=\sqrt{a}$.

Transposing, $\sqrt{x+a}=\sqrt{a}+\sqrt{x-a}$;

squaring, $x+a=a+2\sqrt{ax-a^2}+(x-a)$;

reducing, $a=2\sqrt{ax-a^2}$;

squaring, $a^2=4(ax-a^2)$;

whence $x=\frac{5a}{4}$.

(13) $\sqrt{x+12}=2+\sqrt{x}$.

Squaring, $x+12=4+4\sqrt{x}+x$;
reducing, $2=\sqrt{x}$,
squaring, $x=4$.

(14) $\sqrt{8+x}=2\sqrt{1+x}-\sqrt{x}$.

Squaring, $8+x=4(1+x)-4\sqrt{x+x^2}+x$,
transposing, and reducing $4\sqrt{x+x^2}=4(x-1)$;
dividing, $\sqrt{x+x^2}=x-1$;
squaring, $x+x^2=x^2-2x+1$;
reducing, $3x=1$;
whence $x=\frac{1}{3}$.

(15) $\sqrt{5x}+\frac{12}{\sqrt{5x+6}}=\sqrt{5x+6}$.

Multiply by $\sqrt{5x+6}$, $\sqrt{25x^2+30x}+12=5x+6$;
transposing, $\sqrt{25x^2+30x}=5x-6$;
squaring, $25x^2+30x=25x^2-60x+36$;
reducing, $90x=36$,
whence, $x=\frac{2}{5}$.

(16) $\sqrt{x}-4=\frac{237-10x}{4+\sqrt{x}}$.

Multiplying by $4+\sqrt{x}$, $x-16=237-10x$;
transposing, $11x=253$;
dividing, $x=23$.

(17) $\sqrt{x^2+\sqrt{4x^2+x+\sqrt{9x^2+12x}}}=1+x$.

Squaring, $x^2+\sqrt{4x^2+x+\sqrt{9x^2+12x}}=1+2x+x^2$;
omitting x^2 on each side, and squaring again, we have

$$4x^2+x+\sqrt{9x^2+12x}=1+4x+4x^2;$$

reducing, $\sqrt{9x^2+12x}=1+3x$;
squaring, $9x^2+12x=1+6x+9x^2$;
reducing, $6x=1$,
and $x=\frac{1}{6}$.

(18) $$\sqrt{a+\sqrt{ax}}=\sqrt{a}-\sqrt{a-\sqrt{ax}}.$$

Squaring, $a+\sqrt{ax}=a-2\sqrt{a^2-a\sqrt{ax}}+(a-\sqrt{ax})$;

transposing and reducing, $2\sqrt{a^2-a\sqrt{ax}}=a-2\sqrt{ax}$;

squaring, $4(a^2-a\sqrt{ax})=a^2-4a\sqrt{ax}+4ax$;

reducing, $3a^2=4ax$,

whence $x=\frac{3a}{4}$.

(19) $$b(\sqrt{x}+\sqrt{b})=a(\sqrt{x}-\sqrt{b}).$$

Transposing, $a\sqrt{x}-b\sqrt{x}=a\sqrt{b}+b\sqrt{b}$;

factoring, $(a-b)\sqrt{x}=(a+b)\sqrt{b}$;

squaring, $(a-b)^2x=(a+b)^2b$;

whence $x=\frac{(a+b)^2b}{(a-b)^2}$.

(20) $$\sqrt{x}+\sqrt{ax}=a-1.$$

Factoring, $\sqrt{x}(1+\sqrt{a})=a-1=(\sqrt{a}+1)(\sqrt{a}-1)$, dividing both sides by $1+\sqrt{a}$, and observing that $1+\sqrt{a}$ is the same as $\sqrt{a}+1$,

$\sqrt{x}=(\sqrt{a}-1)$;

squaring, $x=(\sqrt{a}-1)^2$.

Article 211.

QUESTIONS PRODUCING INCOMPLETE EQUATIONS OF THE SECOND DEGREE

(2) Let $x=$ the number.

Then $\frac{x}{3}\times\frac{x}{4}=108$;

whence $\frac{x^2}{12}=108$;

$x^2=1296$;

$x=36$.

(3) Let $x=$ the number.

Then $x^2-16=\frac{x^2}{2}+16$;

whence $\frac{x^2}{2}=32$, and $x=8$.

(4) Let $x=$ the number.

Then $x^2-54=\left(\frac{x}{2}\right)^2+54$;

$x^2-54=\frac{x^2}{4}+54$;

transposing, $\frac{3x^2}{4}=108$; from which $x=12$.

(5) Let $x=$ the number.

Then $\frac{x}{9}=\frac{16}{x}$;

multiply by $9x$ to remove the denominators,

$x^2=9\times16$;

$x=3\times4=12$.

(7) Let $3x$ and $4x$ represent the numbers.
Then $16x^2-9x^2=63$;
from which $7x^2=63$, and $x=3$.
Hence, $3x=9$, and $4x=12$.

(8) Let $3x$ and $4x$ represent the numbers.
Then $9x^2+16x^2=100$;
from which $x^2=4$, and $x=2$.
Hence, $3x=6$, and $4x=8$.

(9) Let $x=$ the number.
Then $(x+3)(x-3)=40$;
$x^2-9=40$;
from which $x^2=49$, and $x=7$.

(10) Let $5x=$ the breadth and $9x=$ the length
Then $5x\times9x=45x^2=$ the number of square feet;
therefore, $45x^2=1620$;
from which $x^2=36$, and $x=6$.
Hence, $5x=30$, and $9x=54$.

(11) Let $10x=$ the number of acres in the farm; then $x=$ the cost per acre in dollars.
Therefore, $10x\times x=10x^2=1000$, the cost of the farm;
from which $x^2=100$, and $x=10$, the cost per acre.
Hence, $10x=100$, the number of acres.

(12) By placing $10x=$ their sum, we have the greater $=7x$, since their sum is to the greater as 10 to 7. And if

$10x=$ their sum, and $7x=$ the greater, the less $=10x-7x=3x$.
Therefore, $10x\times 3x=30x^2=270$;
from which $x^2=9$, and $x=3$.
Hence, $7x=21$, and $3x=9$.

(13) Let $2x=$ their difference, then $9x$ will be the greater, and $9x-2x=7x$, will be the less.
Therefore, $(9x)^2-(7x)^2=128$;
$81x^2-49x^2=128$;
from which $x^2=4$, and $x=2$.
Hence, $9x=18$, and $7x=14$.

(14) Let $x=$ the cost of an orange, then $3x=$ the number of oranges.
Then $3x\times x=3x^2=48$;
from which $x=4$, the cost of an orange;
and $3x=12$, the number of oranges.

(15) Let $4x=$ the cost of 1 yard in cents, then $9x=$ the number of yards.
Then $9x\times 4x=36x^2=324$;
from which $x^2=9$, and $x=3$.
Hence, $4x=12$, the cost per yard;
and $9x=27$, the number of yards.

(16) Let $\frac{1}{2}x$ and $\frac{2}{3}x$ represent the numbers.
Then $\frac{1}{4}x^2+\frac{4}{9}x^2=225$;
multiplying by 4×9, to remove fractions, we have
$9x^2+16x^2=225\times 4\times 9$;
$25x^2=225\times 4\times 9$;
dividing $x^2=9\times 4\times 9$;
extracting the sq. root, $x=3\times 2\times 3=18$.
Hence, $\frac{1}{2}x=9$, and $\frac{2}{3}x=12$.

We may avoid fractions by representing the numbers by $3x$ and $4x$, as recommended in the book.

(17) By reducing $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$, to a common denominator, we find they are to each other as 6, 8, and 9; therefore, let the three numbers be represented by $6x$, $8x$, and $9x$.
Then $36x^2+64x^2+81x^2=724$;
adding, $181x^2=724$;
from which $x^2=4$, and $x=2$.
Hence, $6x=12$, $8x=16$, and $9x=18$.

(18) Let $4x$ = the price of a yard, then $5x$ = the number of yards.

Then $20x^2$ = whole cost;

and $\frac{20x^2+45}{5x}$ = cost of a yard if he had received 45 cents more for the same piece.

Therefore, $\frac{20x^2+45}{5x} : 5x :: 5 : 4$

whence, $\frac{80x^2+180}{5x}=25x$;

multiplying by $5x$, $80x^2+180=125x^2$;

from which $x^2=4$, and $x=2$.

Hence, $4x=8$, and $5x=10$.

Article 212.

COMPLETE EQUATIONS OF THE SECOND DEGREE.

(50) $2ax-x^2=-2ab-b^2$.

$x^2-2ax=2ab+b^2$, by changing the signs,

$x^2-2ax+a^2=a^2+2ab+b^2$, by completing the square;

$x-a=\pm(a+b)$, by extracting the sq. root;

transposing, $x=a\pm(a+b)=2a+b$, or $-b$.

(51) $x^2-2ax=b^2-a^2$.

$x^2-2ax+a^2=b^2$, by completing the square;

$x-a=\pm b$, by extracting the square root;

$x=a\pm b=a+b$, or $a-b$.

(52) $x^2+3bx-4b^2=0$.

$x^2+3bx=4b^2$, by transposing;

$x^2+3bx+\frac{9b^2}{4}=4b^2+\frac{9b^2}{4}=\frac{25b^2}{4}$, by completing the sq.;

$x+\frac{3b}{2}=\pm\frac{5b}{2}$, by extracting the square root;

$x=-\frac{3b}{2}\pm\frac{5b}{2}=+b$, or $-4b$.

(53) $x^2-ax-bx=-ab$.

$x^2-(a+b)x=-ab$, by factoring;

$x^2-(a+b)x+\frac{(a+b)^2}{4}=\frac{(a+b)^2}{4}-ab=\frac{a^2-2ab+b^2}{4}$;

$$x-\frac{a+b}{2}=\pm\frac{a-b}{2};$$

$$x=\frac{a+b}{2}\pm\frac{a-b}{2}=+a, \text{ or } +b.$$

(54) $\frac{x}{x+a}=\frac{b}{x-b}$.

$x^2-bx=bx+ab$, by clearing of fractions;

$x^2-2bx=ab$;

$x^2-2bx+b^2=ab+b^2$;

$x-b=\pm\sqrt{ab+b^2}$;

$x=b\pm\sqrt{ab+b^2}$.

(55) $2bx^2+(a-2b)x=a$.

$$x^2+\frac{a-2b}{2b}x=\frac{a}{2b};$$

$$x^2+\frac{a-2b}{2b}x+\frac{(a-2b)^2}{16b^2}=\frac{(a-2b)^2}{16b^2}+\frac{a}{2b}=\frac{a^2+4ab+4b^2}{16b^2};$$

$$x+\frac{a-2b}{4b}=\pm\frac{a+2b}{4b};$$

$$x=-\frac{a-2b}{4b}\pm\frac{a+2b}{4b}=1, \text{ or } -\frac{a}{2b}.$$

(56) $\frac{x^2}{a^2}-\frac{x}{b}=\frac{2a^2}{b^2}$.

$x^2-\frac{a^2}{b}x=\frac{2a^4}{b^2}$, by multiplying by a^2;

$$x^2-\frac{a^2}{b}x+\frac{a^4}{4b^2}=\frac{a^4}{4b^2}+\frac{8a^4}{4b^2}=\frac{9a^4}{4b^2}=\left(\frac{3a^2}{2b}\right)^2;$$

$$x-\frac{a^2}{2b}=\pm\frac{3a^2}{2b};$$

$$x=\frac{a^2}{2b}\pm\frac{3a^2}{2b}=\frac{2a^2}{b}, \text{ or } -\frac{a^2}{b}.$$

(57) $x^2-(a-1)x-a=0$.

$x^2-(a-1)x=a$;

$$x^2-(a-1)x+\frac{(a-1)^2}{4}=\frac{(a-1)^2}{4}+\frac{4a}{4}=\frac{(a+1)^2}{4};$$

$$x-\frac{a-1}{2}=\pm\frac{a+1}{2};$$

$$x=\frac{a-1}{2}\pm\frac{a+1}{2}=a, \text{ or } -1.$$

(58) $x^2-(a+b-c)x=(a+b)c.$

$$x^2-(a+b-c)x+\frac{(a+b-c)^2}{4}=\frac{(a+b-c)^2}{4}+\frac{4(a+b)c}{4}$$

$$=\frac{a^2+2ab+b^2-2ac-2bc+c^2+4ac+4bc}{4}$$

$$=\frac{a^2+2ab+b^2+2ac+2bc+c^2}{4}=\frac{(a+b+c)^2}{4};$$

by extracting the square root of both sides, we find

$$x-\frac{a+b-c}{2}=\pm\frac{a+b+c}{2};$$

$$x=\frac{a+b-c}{2}\pm\frac{a+b+c}{2}=a+b, \text{ or } -c.$$

Article 214.

PROBLEMS PRODUCING COMPLETE EQUATIONS OF THE SECOND DEGREE.

(6) Let $x=$ the number.
Then $x^2-6x=7$.
from which $x=+7$, or -1.

The positive value satisfies the given question in an arithmetical sense, and the negative value satisfies the following question in an arithmetical sense.

Find a number, such that if its square be *increased* by 6 times the number itself, the *sum* shall be 7.

(7) Let $x=$ the number.
Then $x^2+8x=9$.
from which $x=+1$, or -9.

The positive value satisfies the question in an arithmetical sense, and the negative value satisfies a question expressed in the same words, except that *increased* should be *diminished*, and *sum* should be *difference*.

(8) Let $x=$ the number.
Then $2x^2+3x=65$;
from which $x=+5$, or $-\frac{13}{2}$.

(9) Let $x=$ the number
Then $\frac{2}{3}(x^2-1)=\frac{5x}{2}$;

from which, $x=+4$, or $-\frac{1}{4}$.

The negative value is the answer, in an arithmetical sense, to the following question.

Find a number such, that if 1 *be diminished by its square*, and $\frac{2}{3}$ of the remainder be taken, the result shall be equal to 5 times the number divided by 2.

(10) Let $x=$ the number.

Then $\frac{44}{x-2}=\frac{1}{4}x-4$;

$44=\frac{1}{4}x^2-4\frac{1}{2}x+8$, by multiplying by $x-2$;

$x^2-18x=144$, by clearing, transposing and changing signs;

whence $x=+24$, or -6.

(11) Let $x=$ the greater number, then $x-8=$ the less.

Therefore $x^2-8x=240$;

from which $x=+20$, or -12.

Hence, $x-8=+12$, or -20.

(12) Let $x=$ the number of sheep.

Then $\frac{80}{x}=$ cost of one;

also, $\frac{80}{x+4}=$cost of one, if he had bought 4 more for the same money.

Therefore $\frac{80}{x+4}+1=\frac{80}{x}$.

$80x+x^2+4x=80x+320$, by clearing of fractions,

$x^2+4x=320$, from which $x=+16$, or -20.

The negative value is the answer, in an arithmetical sense to the following question.

A person bought a number of sheep for 80 dollars; if he had bought 4 *less* for the same money, he would have paid 1 dollar *more* for each; how many did he buy?

(13) Let $x=$ the greater number, then $x-10=$ the less.

Then $\frac{600}{x-10}-\frac{600}{x}=10$.

$600x-600x+6000=10x^2-100x$; by clearing of frct'ns,

whence $x^2-10x=600$;

from which $x=+30$, or -20,

and $x-10=+20$, or -30.

(14) Let $x=$ the rate of travel.

Then $\frac{45}{x}=$ number of hours traveling at x miles per hour;

and $\frac{45}{x+\frac{1}{2}}=$ number of hours traveling at $(x+\frac{1}{2})$ miles per hour.

Therefore, $\frac{45}{x+\frac{1}{2}}+1\frac{1}{4}=\frac{45}{x}$;

from which we find $x=+4$, or $-4\frac{1}{2}$.

The negative value is the answer, in an arithmetical sense, to a question expressed in the same words, except that *increases* should be *diminishes*, and *sooner* should be *later*.

(15) Let $x=$ one of the numbers, then $14-x=$ the other.

Then $x^2+(14-x)^2=100$;

$x^2+196-28x+x^2=100$;

reducing, $x^2-14x=-48$;

from which $x=7\pm1=+8$, or $+6$;

and $14-x=+6$, or $+8$.

(16) Let $x=$ the number of rows, then $x+5=$ number of trees in a row,

and $x(x+5)=$ the whole number of trees.

Therefore $x^2+5x=204$;

from which $x=+12$, or -17;

and $x+5=+17$, or -12.

The negative value is the answer to a similar question, the word *more* being changed to *less*.

(17) Let $x=$ the age of the boy, then $x-4=$ his sister's age.

Then $2(x-4)^2+7=x^2$;

from which $x=+13$, or $+3$,

and $x-4=+9$, or -1.

(18) Let $x=$ B's rate of travel; then $x+3=$ A's rate.

Then $\frac{150}{x}-\frac{150}{x+3}=8\frac{1}{3}$;

$150x+450-150x=8\frac{1}{3}x^2+25x$; by clearing of fractions;

reducing $x^2+3x=54$;

from which $x=+6$, or -9,

and $x+3=+9$, or -6.

(19) Let $x=$ the number in the company at first,

then $\frac{175}{x}=$ what each ought to have paid,

and $\frac{175}{x-2}=$ what those paid who remained.

Therefore $\frac{175}{x-2}-\frac{175}{x}=10.$

$175x-175x+350=10x^2-20x$, by clearing of frct'ns ;

$x^2-2x=35$, by reducing ;

from which $x=+7$, or -5.

(20) Let $x=$ the larger number ; then $\frac{100}{x}=$ the smaller.

Therefore, $(x-1)\left(\frac{100}{x}+1\right)=120$;

$\frac{(x-1)(100+x)}{x}=120$;

$100x+x^2-100-x=120x$;

$x^2-21x=100$;

from which $x=+25$, or -4 ;

and $\frac{100}{x}=+4$, or -25.

(21) Let $x^2=$ the father's age ; then $\frac{x^2-4}{3}=$ the son's age.

Then $\frac{1}{2}\left(\frac{x^2-4}{3}-1\right)=x.$

$\frac{x^2-4}{3}-1=2x$, by multiplying by 2,

$x^2-4-3=6x$, by multiplying by 3,

$x^2-6x=7$, by transposing ;

from which $x=+7$, or -1

Hence, $x^2=49$, and $\frac{x^2-4}{3}=15$.

(22) Let $x^2=$ her age.

Then $\frac{3x^2}{8}+x=10$;

from which $x=+4$, or $-\frac{20}{3}$.

Hence, $x^2=16$, or $44\frac{4}{9}$, the former of which satisfies the conditions of the question in its arithmetical sense.

(23) Let $x^2=$ the number.

Then $x^2-\frac{3x}{5}=22$;

from which $x=+5$, or $-\frac{22}{5}$;

Hence, $x^2=25$, or $19\frac{9}{25}$, the former of which satisfies the conditions of the question in its arithmetical sense.

(24) Let $x=$ the number of yards.

Then $\frac{600}{x}=$ cost per yard in cents,

and $\frac{540}{x-15}=$ selling price per yard in cents.

Therefore $\frac{600}{x}+1=\frac{540}{x-15}$;

from which $x=+75$, or -120.

Hence, $\frac{600}{x}=+8$, or -5.

The negative values are the answer, in an arithmetical sense to the following question.

A merchant bought a piece of muslin for 6 dollars: after *adding* to it 15 yards, he sold the whole for 5 dollars and 40 cents; at which rate he received 1 cent a yard *less* than the piece cost him; how many yards did he buy, and at what price?

(25) Let $x=$ cost, then $\frac{x}{100}=$ per cent of loss,

and $x\times\frac{x}{100}=\frac{x^2}{100}=$ loss.

Therefore $x-\frac{x^2}{100}=24$,

whence $x=+60$, or $+40$.

Article 219.

EQUATIONS OF THE SECOND DEGREE, CONTAINING TWO UNKNOWN QUANTITIES.

(6) $x^2+y^2=34$, (1);
$x^2-y^2=16$, (2).

By adding these equations together, and dividing by 2, we find $x^2=25$; from which $x=+5$, or -5.

The value of y may be found either by substituting 25 instead of x^2, or by subtracting the second equation from the first.

(7) $x+y=16$, (1);
$xy=63$, (2).

The values of x and y are readily obtained by finding the value of either in terms of the other from equation (2), and substituting it in equation (1); or thus:

$x^2+2xy+y^2=256$, by squaring (1);
$4xy=252$ by multiplying (2) by 4;

$x^2-2xy+y^2=4$, by subtracting;
$x-y=2$, by extracting the square root;
$x+y=16$.

From these equations, by adding and subtracting, the values of x and y are readily found.

(8) $x-y=5$, (1);
$xy=36$, (2).

From Eq. (1) $y=x-5$; this being substituted instead of y in Eq. (2) gives $x^2-5x=36$, from which x is readily found, and then y.

Or, by squaring Eq. (1), then adding it to 4 times Eq. (2), and extracting the square root, we find $x+y=11$; from which, and Eq. (1), by adding and subtracting, the values of x and y are readily found.

(9) $x+y=9$, (1);
$x^2+y^2=53$, (2).

From Eq. (1), $y=9-x$; this being substituted instead of y in Eq. (2) gives, after reducing, $x^2-9x=-14$; from which we find $x=7$, or 2; consequently $y=2$, or 7.

(10) $x-y=5$, (1);
$x^2+y^2=73$, (2).

From Eq. (1) $y=x-5$; this being substituted instead of y in Eq. (2), gives, after reducing, $x^2-5x=24$; from which x is found $=8$, or -3; hence $y=3$, or -8.

(11) $x^3+y^3=152$, (1);
$x+y=8$, (2).

Dividing Eq. (1) by Eq. (2), we find $x^2-xy+y^2=19$, (3).

From Eq. (2) $y=8-x$; substituting this value of y in Eq. (3) and reducing, we have $x^2-8x=-15$; from which we find $x=5$, or 3; hence $y=3$, or 5.

(12) $x^3-y^3=208$, (1);
$x-y=4$, (2).

Dividing Eq. (1) by Eq. (2), we find $x^2+xy+y^2=52$, (3).

From Eq. (2), $y=x-4$; substituting this value of y in Eq. (3) and reducing, we find $x^2-4x=12$; from which $x=6$, or -2: hence $y=2$, or -6.

(13) $x^3+y^3=19(x+y)$, (1);
$x-y=3$, (2).

By dividing both sides of Eq. (1) by $x+y$,
$x^2-xy+y^2=19$, (3).

From Eq. (2) $y=x-3$; substituting this value of y in Eq. (3) and reducing, we find $x^2-3x=10$; from which $x=5$, or -2; hence $y=2$, or -5.

(14) $x+y=11$, (1);
$x^2-y^2=11$, (2).

From Eq. (1) $y=11-x$; this being substituted in Eq. (2) instead of y, and the equation reduced, gives $22x=132$, from which $x=6$; hence $y=5$.

(15) $(x-3)(y+2)=12$, (1);
$xy \quad =12$, (2).

Performing the operations indicated in Eq. (1) and then subtracting Eq. (2) from it, we find $2x-3y=6$, (3). From Eq. (3) we find $x=\frac{6+3y}{2}$, and this being substituted in Eq. (2), gives, after reducing, $y^2+2y=8$; from which $y=2$, or -4; hence $x=6$, or -3

(16) $y-x=2$, (1);
$3xy=10x+y$, (2).

From Eq. (1) $y=x+2$; this value of y being substituted in Eq. (2), gives, after reducing, $3x^2-5x=2$; from which $x=2$, or $-\frac{1}{3}$; hence, $y=4$, or $1\frac{2}{3}$.

(17) $3x^2+2xy=24$, (1);
$5x-3y=1$, (2).

From Eq. (2) $y=\frac{5x-1}{3}$; this being substituted in Eq. (1), gives, after reducing, $19x^2-2x=72$; from which $x=2$, or $-\frac{36}{19}$; hence, $y=3$, or $-\frac{199}{57}$.

(18) $\frac{1}{x}+\frac{1}{y}=\frac{5}{6}$, (1);

$\frac{1}{x^2}+\frac{1}{y^2}=\frac{13}{36}$, (2).

Let $\frac{1}{x}=v$, and $\frac{1}{y}=z$; then $v+z=\frac{5}{6}$, (3);

and $v^2+z^2=\frac{13}{36}$, (4).

From Eq. (3) $z=\frac{5}{6}-v=\frac{5-6v}{6}$; substituting this value instead of z in Eq. (4) and reducing, we find

$6v^2-5v=-1$; from which $v=\frac{1}{2}$, or $\frac{1}{3}$; and substituting in the equation, $z=\frac{5}{6}-v$, we find $z=\frac{1}{3}$, or $\frac{1}{2}$.

Hence, $v=\frac{1}{x}=\frac{1}{2}$, or $\frac{1}{3}$; from which $x=2$, or 3.

$z=\frac{1}{y}=\frac{1}{3}$, or $\frac{1}{2}$; from which $y=3$, or 2.

(19) $x-y=2$, (1).

$x^2y^2=21-4xy$, (2).

In Eq. (2) let $xy=z$, the equation then becomes

$z^2=21-4z$; from which z or $xy=3$, or -7.

We then have $x-y=2$

$xy=3$, to find x and y.

These equations are similar to those in example 8, and we readily find $x=3$, or -1; hence $y=1$, or -3.

From the equations $x-y=2$ and $xy=-7$, we may also find two other values of x and y, but they are imaginary.

Article 219.

PROBLEMS PRODUCING EQUATIONS OF THE SECOND DEGREE, CONTAINING TWO UNKNOWN QUANTITIES.

(1) Let x and y represent the numbers.

Then $x+y=10$, (1);

$x^2+y^2=52$, (2).

Solved like question 9, preceding.

(2) Let x and y represent the numbers.

Then $x-y=3$, (1);

$x^2-y^2=39$, (2).

Divide Eq. (2) by Eq. (1) and we get $x+y=13$, (3); then from this and Eq. (1), we readily find $x=8$, and $y=3$.

(3) Let x^2 and y^2 represent the parts.

Then $x^2+y^2=25$, (1);

$x+y=7$, (2).

The values of x and y may now be found in the same manner as in question 9, preceding.

(4) Let $x=$ the digit in ten's place, and $y=$ the digit in unit's place.

Then $10x+y=$ the number;

$(10x+y)(x+y)=160$, (1);

$\frac{10x+y}{4y}=4$, (2).

Dividing Eq. (1) by Eq. (2) we get

$$4y(x+y)=40\text{; from which } x=\frac{10-y^2}{y}.$$

Substituting this instead of x, in equation 2, we get

$$10\left(\frac{10-y^2}{y}\right)+y=16y;$$

clearing of fractions, $100-10y^2+y^2=16y^2$;

from which $y=2$; hence $x=3$.

(5) Let $x=$ the greater number, and $y=$ their difference, then $x-y=$ the less.

Then $xy=16$, (1);

and $xy-y^2=12$, (2).

Subtracting the 2d equation from the 1st, we get

$y^2=4$; hence $y=2$;

$x=\frac{16}{y}=8$, and $x-y=6$.

(6) Let x and y represent the numbers.

Then $x+y=10$, (1);

$xy-(x-y)=22$, (2).

Find the value of either x or y from Eq. (1), and substitute it instead of the same unknown quantity, in Eq. (2).

This may also be easily solved by means of one unknown quantity; thus, let $x=$ one of the parts, then $10-x=$ the other, and $10-2x=$ their difference.

Then $x(10-x)-(10-2x)=22$;
from which $x=8$, or 4;
hence $10-x=2$, or 6.

The numbers 4 and 6, satisfy the conditions of the question in an arithmetical sense. The numbers 8 and 2 satisfy the following problem. Divide 10 into two such parts, that their product *plus* their difference, may be 22.

(7) Let x and y represent the numbers.
Then $x+y=10$, (1);
$x^3+y^3=370$, (2).

For the method of solution, see question 11, preceding.

(8) Let x and y represent the numbers.
Then $x-y=2$, (1);
$x^3-y^3=98$, (2).

For the method of solution, see question 12, preceding.

(9) Let $x=$ the greater, and $y=$ the less of the two numbers.
Then $6x+5y=50$, (1);
$xy=20$, (2).

The value of y, from Eq. (2), is $\frac{20}{x}$; this being substituted in Eq. (1) and reduced, gives
$6x^2-50x=-100$; from which
$x=5$, or $3\frac{1}{3}$;
hence, $y=4$, or 6.

The first values of x and y satisfy the question; the other two satisfy a question precisely similar, except that the words *greater* and *less* are transposed.

(10) Let $x=$ the digit in ten's place, and $y=$ the digit in unit's place.
Then $10x+y=$ the number.
$\frac{10x+y}{xy}=2$, (1);
$10x+y+27=10y+x$, (2).

From the 2d equation $y=x+3$; this being substituted in the 1st equation, we get, after reducing,
$2x^2-5x=3$; from which we find
$x=3$; hence $y=6$, and the number is 36.

(11) Let x, y, and z represent the numbers.

Then $\frac{xy}{z}=a$, (1);

$\frac{xz}{y}=b$, (2);

$\frac{yz}{x}=c$, (3).

Multiply the three equations together, and we have

$xyz=abc$, (4).

Divide this successively by each of the equations (1), (2), and (3), and we obtain

$z^2=bc$, $y^2=ac$, and $x^2=ab$.

Hence, $x=\pm\sqrt{ab}$, $y=\pm\sqrt{ac}$, and $z=\pm\sqrt{bc}$.

(12) Let x and y represent the numbers.
Then $x+y=9$, (1);
$x^3+y^3=21(x+y)$, (2).
Divide both sides of Eq. (2) by $x+y$, and we get
$x^2-xy+y^2=21$, (3).
The value of y, from Eq. (1), is $9-x$; this being substituted in Eq. (3), we have, after reducing,
$x^2-9x=-20$, from which
$x=5$, or 4; hence $y=4$, or 5.

(13) Let $x+y$, and $x-y$ represent the numbers.
Then the sum of their squares $=2x^2+2y^2$;
the difference of their squares $=4xy$;
and their product $=x^2-y^2$.
Therefore $2x^2+2y^2-2(x^2-y^2)=4$, (1),
$4xy-\frac{1}{2}(x^2-y^2)=4$, (2).

Reducing Eq. (1) we readily find $y=1$; this value being substituted in Eq. (2), we have, after reduction,

$x^2-8x=-7$, from which $x=7$.

Hence, $x+y=8$; and $x-y=6$.

(14) Let $x=$ the circumference of the less wheel, and $y=$ the circumference of greater.

Then $\frac{120}{x}=\frac{120}{y}+6$, (1);

$\frac{120}{x+1}=\frac{120}{y+1}+4$, (2).

$120y=120x+6xy$, by clearing Eq. (1) of fractions;

$120y+120=120x+120+4xy+4x+4y+4$, by clearing Eq. (2) of fractions.

$116y=124x+4xy+4$, by reducing.

From Eq. (1) after clearing, we find $y=\frac{20x}{20-x}$; this being substituted in the last equation, gives, after clearing of fractions and reducing,

$11x^2-39x=20$; from which

$x=4$; hence $y=5$.

(15) Let x and y represent the rates of travel of A and B; then $2x=$ distance A travels in 2 hours at x miles per hour,

and $2x+1=$ distance A travels in 2 hours at $x+\frac{1}{2}$ miles per hour.

$30-2x=$ distance A travels after B starts, in 1st case;

$42-(2x+1)=41-2x=$ distance A travels after B starts, in 2d case.

Therefore $\frac{30-2x}{x}=\frac{30}{y}$, (1);

$\frac{41-2x}{x+\frac{1}{2}}=\frac{42}{y+\frac{1}{2}}$, (2).

Clearing Eq. (1) of fractions, and reducing, we find

$$y=\frac{15x}{15-x}.$$

Clearing Eq. (2) of fractions, and reducing, we get

$41y-2xy=43x+\frac{1}{2}$.

Substituting the value of y before found, in this equation clearing of fractions, and reducing, we get

$26x^2-59x=15$;

from which $x=2\frac{1}{2}$; hence $y=3$.

(16) Let $x=$ the number of miles B traveled;

then $x+30=$ the number of miles A traveled.

Then since the distance traveled, divided by the number of days spent in traveling, gives the number of miles traveled per day,

$$\frac{x}{4}=\text{ A's rate of travel;}$$

$$\frac{x+30}{9}=\text{ B's rate of travel.}$$

Then dividing the distance traveled by each man's rate of travel,

$$(x+30)\div\frac{x}{4}=\frac{4(x+30)}{x}=\text{days A traveled.}$$

$$x\div\frac{x+30}{9}=\frac{9x}{x+30}=\text{days B traveled.}$$

But they both traveled the same number of days, therefore,

$$\frac{4(x+30)}{x}=\frac{9x}{x+30};$$

$4(x+30)^2=9x^2$, by clearing of fractions;

$2(x+30)=3x$, by extracting the square root;

from which $x=60$; hence $x+30=90$,

and $60+90=150$ miles, the distance from A to B.

ARITHMETICAL PROGRESSION.

Article 222.

(11) Here $n=20$, $a=16\frac{1}{12}$, $d=48\frac{1}{4}-16\frac{1}{12}=32\frac{1}{6}$.

$l=a+(n-1)d=16\frac{1}{12}+(20-1)32\frac{1}{6}=16\frac{1}{12}+611\frac{1}{6}=627\frac{1}{4}$

Article 223.

(5) Here $l=a+(n-1)d=10-3\times9=-17$.

$$s=(l+a)\frac{n}{2}=(-17+10)\frac{10}{2}=-35.$$

Article 225.

EXAMPLES.

(1) $s=(l+a)\frac{n}{2}=(1+1000)\frac{1000}{2}=500500.$

(2) $l=a+(n-1)d=1+(101-1)2=201.$

$$s=(l+a)\frac{n}{2}=(201+1)\frac{101}{2}=10201.$$

(3) First find how many times a clock strikes in 12 hours Here $a=1$, $l=12$, $n=12$,

$$s=(12+1)\frac{12}{2}=78.$$

$78\times 2=156=$ strokes per day;
$156\times 7=1092=$ strokes in a week.

(4) Since the second term is 2, the 3d term 3, and so on, the nth term is evidently n.
Or thus, $l=a+(n-1)d=1+(n-1)1=1+n-1=n$.

$$s=(l+a)\frac{n}{2}=(n+1)\frac{n}{2}=\frac{1}{2}n(n+1).$$

(5) Here $d=2$; $l=1+(n-1)2=1+2n-2=2n-1$.

$$s=(l+a)\frac{n}{2}=(2n-1+1)\frac{n}{2}=n^2.$$

(6) Substituting the values of l, a, and d, in the formula, $l=a+(n-1)d$, we have
$29=2+(n-1)3$; from which $n=10$.

$$s=(l+a)\frac{n}{2}=(29+2)\frac{10}{2}=155.$$

(7) Substituting the values of l, a, and n, in the formula, $l=a+(n-1)d$, we have
$10=6+(9-1)d$; from which $d=\frac{1}{2}$.

$$s=(l+a)\frac{n}{2}=(10+6)\frac{9}{2}=72.$$

(8) Substituting the values of s, a, and n, in the formula,

$s=(l+a)\frac{n}{2}$, we have

$85=(l+10)\frac{10}{2}$; from which $l=7$.

Substituting the values of l, a, and n, in the formula, $l=a-(n-1)d$, we have
$7=10-(10-1)d$; from which $d=\frac{1}{3}$.

(9) Substituting the values of a, b, and m, in the formula, $d=\frac{b-a}{m+1}$, we have $d=\frac{16-1}{4+1}=3$.
Hence the series is 1, 4, 7, 10, 13, 16, &c.

(10) Substituting the values of a, and d, in the formula, $l=a+(n-1)d$, we have $l=24-4(n-1)=28-4n$; substituting this value of l, and those of s and a in the formula,

$s=(l+a)\frac{n}{2}$, we have $72=(28-4n+24)\frac{n}{2}$; from which, by reducing, $n^2-13n=-36$.

From this equation $n=+9$, or $+4$.

(11) Let $n=$ the number of acres; then the nth acre evidently cost n dollars.

Substituting n for l in the formula, $s=(l+a)\frac{n}{2}$, and for s and a their values, we get

$$12880=(n+1)\frac{n}{2};\ \text{by reducing}$$

$$n^2+n=25760;\ \text{from which } n=160.$$

Having the number of acres, the average price per acre is easily found.

(12) Let $x=$ the number of days; then on the xth day, A will travel xa, or ax miles. Hence, to find the sum of the series, we have the first term $a=a$, $d=a$, and $l=ax$; substituting these in the formula, $s=(l+a)\frac{n}{2}$, we have

$$s=(ax+a)\frac{x}{2}=\frac{1}{2}ax(x+1).$$

Then in $(x-4)$ days, B travels $9a(x-4)$ miles;

Therefore $\frac{1}{2}ax(x+1)=9a(x-4)$; reducing,

$$x^2-17x=-72;\ \text{from which } x=8, \text{ or } 9.$$

(13) Let $n=$ the number of hours.

Then $l=a+(n-1)d=5+(n-1)1=4+n$;

$$s=(l+a)\frac{n}{2}=(4+n+5)\frac{n}{2}=\frac{n}{2}(9+n).$$

Therefore $\frac{n}{2}(9+n)=6(3\frac{1}{3}+n)$.

By reducing $n^2-3n=40$; from which $n=8$.

(14) Let $x=$ the number of hours; then the formula,

$$l=a-(n-1)d \text{ becomes } l=4-\frac{1}{2}(x-1)=4\frac{1}{2}-\frac{1}{2}x;$$

$s=(l+a)\frac{n}{2}$, becomes $s=(4\frac{1}{2}-\frac{1}{2}x+4)\frac{x}{2}=(8\frac{1}{2}-\frac{1}{2}x)\frac{x}{2}$:
but in x days, A travels $3x$ miles;
therefore $3x=(8\frac{1}{2}-\frac{1}{2}x)\frac{x}{2}$; dividing both sides by x and reducing, we find $x=5$.

GEOMETRICAL PROGRESSION.

Article 229.

(4) In this example the ratio is $-\frac{1}{3}$.

$$s=\frac{a}{1-r}=\frac{1}{1-(-\frac{1}{3})}=\frac{1}{\frac{4}{3}}=\frac{3}{4}.$$

(5) Here the ratio is $\frac{1}{x^2}$.

$$s=\frac{a}{1-r}=\frac{1}{1-\frac{1}{x^2}}=\frac{x^2}{x^2-1}.$$

(6) Here the ratio is $-\frac{b}{a}$.

$$s=\frac{a}{1-r}=\frac{a}{1-\left(-\frac{b}{a}\right)}=\frac{a}{1+\frac{b}{a}}=\frac{a^2}{a+b}.$$

(7) Here the ratio is $\frac{1}{2}$.

$$s=\frac{a}{1-r}=\frac{10}{1-\frac{1}{2}}=\frac{10}{\frac{1}{2}}=20.$$

AMBIGUOUS AND ERRONEOUS EXPRESSIONS.

As this work is intended especially for the assistance of young teachers it is thought proper, in conclusion, to call attention to some loose and inaccurate expressions, that are occasionally used in the school room, and which are also to be found in some of the works intended for text books.

Too much importance cannot be attached to clearness and propriety of expression. Accuracy of style has a tendency to produce accuracy of thought. Every definition should be expressed in language the most precise, brief, and clear, of which it is susceptible ; while all explanations and directions, whether contained in the text book or delivered by the teacher, should be given in such a manner, that the pupil cannot possibly mistake the meaning.

Lest some should regard matters of this kind as unworthy of notice, it is proper to add that great attention is paid to them by the French Mathematicians ; hence, many of their works exhibit a perspicuity and simplicity, and a logical clearness of arrangement, which add greatly to their value.

"Place the two quantities under each other." This is not possible, *one* of the quantities may be under the other, but *each* cannot be under the other at the same time. It should be specified which is to be placed below the other.

"Subtract the numerators from each other." It should be specified which is to be taken from the other.

"Find the difference of $x^2—a^2$ and a^2." When numbers are referred to, this expression is correct, but in the case here presented, the difference is either $x^2—2a^2$, or $2a^2—x^2$. It should always be specified which of the two quantities is to be subtracted.

"Divide the numerators by each other, if *they* will exactly divide." This expression has no clear meaning, and the word divide at the close of the sentence is used improperly.

"The two first numbers." "The three first numbers," &c. These expressions are frequently used. When two or more things are considered in regard to order, only *one* can properly be called first ; hence, there is no such thing as the *two* first. However, we can with propriety say "the first two," because there may be a second two, a third two, and so on.

"Neither the first nor the last terms are squares." This should be, "neither the first nor the last term *is a square.*"

"This value is the greatest of all others." Here *others* ought to be omitted, or it might be, "this value is greater than any other."

"An equation of the second degree or power." Equations are of different degrees, but no equation is of the second or any other power.

These examples might be greatly extended. The preceding are given merely as specimens. Such expressions confuse the mind of the pupil and often prevent a clear and accurate understanding of the subject under examination. They cannot, therefore, be regarded with indifference by any one who aspires to the character of an accomplished teacher.

KEY

TO

RAY'S ALGEBRA, PART SECOND.

☞ The numbers in parentheses, as seen in the margin, refer to the corresponding number of example, under the same article in the Algebra.

GREATEST COMMON DIVISOR.

Article 108.

NOTE.—This article contains the first examples in the Algebra which the attentive student will find any real difficulty in solving.

(5)

$$\begin{array}{l|l} a^4-x^4 & a^3+a^2x-ax^2-x^3 \\ \underline{a^4+a^3x-a^2x^2-ax^3} & a+1 \\ -a^3x+a^2x^2+ax^3-x^4 & \\ =-x(a^3-a^2x-ax^2+x^3) & \\ a^3-a^2x-ax^2+x^3 & \\ \underline{a^3+a^2x-ax^2-x^3} & \\ -2a^2x \quad +2x^3 & \\ -2x(a^2-x^2) & \end{array}$$

After dividing we find the first remainder contains a factor, $-x$, not found in the divisor, hence it should be canceled. See Note 3.

By dividing $a^3+a^2x-ax^2-x^3$ by a^2-x^2 we find there is no remainder, hence the latter is the greatest common divisor required.

(6)

$$\begin{array}{l|l} x^3-5x^2+13x-9 & x^3-2x^2+4x-3 \\ \underline{x^3-2x^2+4x-3} & 1 \\ -3x^2+9x-6=-3(x^2-3x+2) & \end{array}$$

$$\begin{array}{l|l} x^3-2x^2+4x-3 & x^2-3x+2 \\ \underline{x^3-3x^2+2x} & x+1 \\ x^2+2x-3 & \\ \underline{x^2-3x+2} & \\ x-1 & \end{array}$$

$x-1$ will be found to divide x^2-3x+2 without a remainder; it is, therefore, the greatest common divisor.

NOTE.— In the solution of the remaining questions in this article, we shall merely exhibit so much of the operation as is necessary to show how the greatest common divisor is obtained. The reasons for the different steps of the operation will be found in the rule, or in the notes following it.

(7)

$$\begin{array}{r|l}
x^3-5x^2+16x-12 & x^3-2x^2-15x+16 \\
x^3-2x^2-15x+16 & 1 \\
\hline
-3x^2+31x-28 & \\
-3x^3+6x^2+45x-48 & -3x^2+31x-28 \\
-3x^3+31x^2-28x & x+25 \\
\hline
-25x^2+73x-48 & \text{. . . Mult. by 3.} \\
-75x^2+219x-144 & \\
-75x+775x-700 & \\
\hline
-556x+556 & \\
-556(x-1) &
\end{array}$$

The 4th line is obtained by multiplying the divisor by -3.

Ans. $x-1$.

(8) Multiplying the first polynomial by 2 to render it divisible by the second, and dividing by x (Note 3), we have

$$\begin{array}{r|l}
42x^2-52x+16 & 6x^2-x-2 \\
42x^2-7x-14 & 7 \\
\hline
-45x+30 & \\
-15(3x-2). &
\end{array}$$

Ans. $3x-2$.

(9)

$$\begin{array}{r|l}
2x^4+11x^3-13x^2-99x-45 & 2x^3-7x^2-46x-21 \\
2x^4-7x^3-46x^2-21x & x+9 \\
\hline
18x^3+33x^2-78x-45 & \\
18x^3-63x^2-414x-189 & \\
\hline
96x^2+336x+144 & \\
48(2x^2+7x+3) &
\end{array}$$

Ans. $2x^2+7x+3$.

(10) Multiplying the first polynomial by 7 to render it divisible by the second, we have

$$\begin{array}{r|l}
7x^4+14x^2+63 & 7x^3-11x^2+15x+9 \\
7x^4-11x^3+15x^2+9x & x+11 \\
\hline
11x^3-x^2-9x+63 & \text{. . . . . Multiply by 7.} \\
77x^3-7x^2-63x+441 & \\
77x^3-121x^2+165x+99 & \\
\hline
114x^2-228x+342 & \\
114(x^2-2x+3) &
\end{array}$$

Ans. x^2-2x+3.

(11) Multiplying the second polynomial by 2, and dividing by the first, we have

$$\begin{array}{l|l} 48x^3-44x^2+34x-10 & 48x^2+16x-15 \\ 48x^3+16x^2-15x & x-5 \\ \hline \quad -60x^2+\ 49x-10 & \\ \quad -240x^2+196x-40 & \\ \quad -240x^2-\ 80x+75 & \\ \hline \qquad\qquad 276x-115=23(12x-5) & \end{array}$$

Ans. $12x-5$.

(12) This example presents no difficulty whatever

(13)
$$\begin{array}{l|l} x^4+a^2x^2+a^4 & x^4+ax^3-a^3x-a^4 \\ x^4+ax^3-a^3x-a^4 & 1 \\ \hline -ax^3+a^2x^2+a^3x+2a^4 & \\ -a(x^3-ax^2-a^2x-2a^3) & \end{array}$$

$$\begin{array}{l|l} x^4+ax^3-a^3x\ -a^4 & x^3-ax^2-a^2x-2a^3 \\ x^4-ax^3-a^2x^2-2a^3x & x+2 \\ \hline a)\ +2ax^3+\ a^2x^2+\ a^3x-\ a^4 & \\ \quad +\ 2x^3+\ a\,x^2+\ a^2x-\ a^3 & \\ \quad +\ 2x^3-\ 2ax^2-2a^2x-4a^3 & \\ \hline +3a)\ +\ 3ax^2+3a^2x+3a^3 & \\ \hline \qquad x^2+ax+a^2 & \end{array}$$

Ans.

(14) In this example $2b$ is a factor of the first polynomial, and $3a$ of the second. Canceling these factors, arranging the terms in both, and multiplying the second by 4, to render it divisible by the first, we have

$$\begin{array}{l|l} 12a^3-12a^2b+4ab^2-4b^3 & 4a^2-5ab+b^2 \\ 12a^3-15a^2b+3ab^2 & 3a+3b \\ \hline \qquad 3a^2b+ab^2-4b^3 & \\ \qquad\qquad\qquad 4 & \\ \hline 12a^2b+\ 4ab^2-16b^3 & \\ 12a^2b-15ab^2+\ 3b^3 & \\ \hline \qquad 19ab^2-19b^3 & \\ \qquad 19b^2(a-b) & \end{array}$$

Ans. $a-b$.

(15)
$$\begin{array}{l|l} x^4-px^3+(q-1)x^2+px-q & x^4-qx^3+(p-1)x^2+qx-p \\ x^4-qx^3+(p-1)x^2+qx-p & 1 \end{array}$$

(*See next page*).

$(q-p)x^3+(q-p)x^2-(q-p)x-(q-p)$,

or x^3+x^2-x-1, by dividing by $q-p$

$$\begin{array}{l|l} x^4-qx^3+(p-1)x^2+qx-p & x^3+x^2-x-1 \\ \underline{x^4+x^3-x^2-x} & x-(q+1) \\ -(q+1)x^3+px^2+(q+1)x-p & \\ \underline{-(q+1)x^3-(q+1)x^2+(q+1)x+(q+1)} & \\ \quad (p+q+1)x^2-(p+q+1) & \\ \quad =(p+q+1)(x^2-1). & \end{array}$$

Ans. x^2-1.

LEAST COMMON MULTIPLE.

Article 113.

(4) From Arts. 85 and 86 it is obvious that $a+x$ is the only divisor of both the quantities. Hence, (Art. 113) $(a^3+x^3)(a^2-x^2)\div(a+x)=(a^3+x^3)(a-x)=a^4-a^3x+ax^3-x^4$. *Ans.*

(5) The quantities separated into their prime factors are $2\times2a(a+x)$, $2\times2\times3x^2(a-x)$, and $3\times3\times2(a+x)(a-x)$; from which we readily see that the least common multiple is $2\times2\times3\times3ax^2(a+x)(a-x)=36ax^2(a^2-x^2)$.

(6) The first quantity divides the second, but not the third, and the second and third have no common factor; therefore, the least common multiple of the three quantities is the product of the second and third.

(7) By examining these quantities we see that the *second* quantity is divisible by the *first*, and the *fourth* by the *third*, and that these are the only cases of divisibility among the four quantities; hence, their least common multiple will be the product of the second and fourth quantities.

(8) By factoring the several quantities, we find the first is $=(x+1)(x-1)$, the $2^{nd}=x^2+1$; $3^{rd}=(x-1)(x-1)$; $4^{th}=(x+1)(x+1)$; $5^{th}=(x-1)(x^2+x+1)$; $6^{th}=(x+1)(x^2-x+1)$, It will now be seen that if we omit the third and fourth quantities the remaining quantities vill contain the factors of these, and no other factor not necessary to be found in the last common multiple. Hence, the l. c. m. will be $(x^2-1)(x^2+1)(x^3-1)(x^3+1)=(x^4-1)(x^6-1)=x^{10}-x^6-x^4+1$.

(9) It is easily seen that 8 is the least common multiple of

the numerical factors, and that of the literal factors, $1-x$, is the only one common to two of them. Hence, the least common multiple is $8(1-x)(1-x)(1+x)(1+x^2)=8(1-x)(1-x^2)(1+x^2)$ $=8(1-x)(1-x^4)$.

(10) We first find the greatest common divisor of the 1st and 2nd polynomials, to be $x-3$; then of the 1st and 3rd to be $3x-2$. Hence,

$$\left.\begin{aligned}3x^2-11x+6&=(x-3)(3x-2)\\2x^2-7x+3&=(x-3)(2x-1)\\6x^2-7x+2&=(2x-1)(3x-2)\end{aligned}\right\}$$

It is evident the least common multiple is $(x-3)(3x-2)(2x-1)$ $=6x^3-25x^2+23x-6$.

TO REDUCE A FRACTION TO ITS LOWEST TERMS.

Article 119.

The only difficulty in solving any of the examples in this article, consists in finding the greatest common divisor of the two terms. In general it may be easily found by the rule (Art. 108) and in most cases by mere inspection. Thus:

(13) From Art. 86 we know that $x+1$ is a divisor of the denominator; and, by trial, it will be found to divide the numerator.

(14) The numerator is the square of $2x-3a$ (Art. 79), and from Art. 83 this will also divide the denominator, since $8x^3-27a^3$ $=(2x)^3-(3a)^3$.

(15) Canceling x in the denominator and multiplying the other factor by 5, we have

$$\begin{array}{r|l}135x^3+315x^2-60x-140 & 15x^3+35x^2+3x+7\\ \underline{135x^3+315x^2+27x+63} & 9\\ -87x-203 & \\ -29(3x+7). & \text{g. c. d.}=3x+7.\end{array}$$

(16) Setting aside the factor 2, which is common to all the terms of the numerator and denominator, as a part of the common divisor, and then multiplying the numerator by 4 to render it divisible by the denominator, the remainder of the operation to find the g. c d. is,

$$\begin{array}{l|l} 4x^3+16x^2y+32xy^2+32y^3 & 4x^2+2xy-12y^2 \\ \underline{4x^3+\ 2x^2y-12xy^2} & \underline{x} \\ \quad 14x^2y+44xy^2+32y^3 & \\ \quad\quad\quad\quad\quad\quad\quad 2 & \\ \overline{\quad 28x^2y+88xy^2+64y^3} & \underline{7y} \\ \quad \underline{28x^2y+14xy^2-84y^3} & \\ \quad\quad\quad 74xy^2+148y^3 & \\ \quad\quad\quad 74y^2(x+2y). & \end{array}$$

$x+2y$ will be found to divide $4x^2+2xy-12y^2$, therefore $2(x+2y)$ is the greatest common divisor of both terms.

(18) $ac+by+ay+bc=(a+b)c+(a+b)y=(a+b)(c+y)$; $af+2bx+2ax+bf=(a+b)f+(a+b)2x=(a+b)(f+2x)$.
Hence $a+b=$ greatest c. d. of both terms.

(19) $6ac+10bc+9ax+15bx=(2c+3x)3a+(2c+3x)5b$ $=(2c+3x)(3a+5b)$.
$6c^2+9cx-2c-3x=(2c+3x)3c-(2c+3x)$ $=(2c+3x)(3c-1)$.
Hence $2c+3x=$ greatest c. d. of both terms.

(20) $x^8+x^6y^2+x^2y+y^3=(x^2+y^2)x^6+(x^2+y^2)y$ $=(x^2+y^2)(x^6+y)$; $x^4-y^4=(x^2+y^2)(x^2-y^2)$.
Hence $x^2+y^2=$ g. c. d. of both terms.

(21) $a^3+(a+b)ax+bx^2=(a^2+bx)a+(a^2+bx)x=(a^2+bx)(a+x)$, $a^4-b^2x^2=(a^2+bx)(a^2-bx)$. Hence $a^2+bx=$ g. c. d.

(22) $ax^m-bx^{m+1}=(ax-bx^2)x^{m-1}=x(a-bx)x^{m-1}$; $a^2bx-b^3x^2$ $=bx(a^2-b^2x^2)=bx(a+bx)(a-bx)$.
Hence $x(a-bx)=$ greatest c. d. of both terms.

(23) $acx^2+(ad+bc)x+bd=ax(cx+d)+b(cx+d)$ $=(ax+b)(cx+d)$;
$a^2x^2-b^2=(ax+b)(ax-b)$. Hence $ax+b=$ g. c. d.

(24) $a^3+ab^2-a^2b-b^3=a(a^2+b^2)-b(a^2+b^2)=(a-b)(a^2+b^2)$.
$4a^4-2a^2b^2-4a^3b+2ab^3=2a^2(2a^2-b^2)-2ab(2a^2-b^2)$ $=(2a^2-2ab)(2a^2-b^2)=2a(a-b)(2a^2-b^2)$.
Hence $a-b$ is the greatest c. d. of both terms.

(25) $2a^2+ab-b^2=a^2+ab+a^2-b^2=a(a+b)+(a+b)(a-b)$ $=(a+b)(a+a-b)=(a+b)(2a-b)$.
$a^3+a^2b-a-b=a^2(a+b)-(a+b)=(a^2-1)(a+b)$.
Hence $a+b$ is the greatest c. d. of both terms.

ART. **120.** Ex. 7.—The following is the operation of finding the greatest common divisor of the two terms:

$$\begin{array}{l|l}
3x^3-3x^2-63x+135 & 3x^2-2x-21 \\
3x^3-2x^2-21x & x-1 \\
\hline
-x^2-42x+135 & \\
\quad 3 & \\
\hline
-3x^2-126x+405 & \\
-3x^2+2x+21 & \\
\hline
-128x+384 & \\
-128(x-3) & x-3= \text{g. c. d.}
\end{array}$$

ADDITION AND SUBTRACTION OF FRACTIONS.

Article 130.

EXAMPLES IN ADDITION OF FRACTIONS.

(12) Reduce the first fraction to the same denominator as the second, by multiplying both terms by $x+a$; the sum of the first two fractions will then be found to be $\frac{2x+5a}{(x+a)^2}$. The sum of this and the third fraction is then readily found by multiplying each numerator by the denominator of the other and taking the sum of the products for the numerator of the result, and the product of the two denominators for the denominator of the result.

(13) The least common multiple of the denominators is readily found to be $4a^3(a+x)(a-x)(a^2+x^2)=4a^3(a^4-x^4)$. We then find for the numerators of the respective fractions, the following quantities:

$$\begin{array}{lll}
1^{st} & (a-x)(a^2+x^2)\ldots\ldots & = a^3+ax^2-a^2x-x^3 \\
2^{nd} & (a+x)(a^2+x^2)\ldots\ldots & = a^3+ax^2+a^2x-x^3 \\
3^{rd} & 2a(a+x)(a-x)\ \ldots\ldots & =2a^3-2ax^2 \\
\hline
 & \text{Sum} & =\frac{4a^3}{4a^3(a^4-x^4)}=\frac{1}{a^4-x^4}. \quad \textit{Ans.}
\end{array}$$

(14) It is most convenient to make the common denominator of the fractions, $abc(a-b)(a-c)(b-c)$. In doing this we must change the signs of the factor, $b-a$, in the denominator of the second fraction, which may be done if at the same time we change the sign of the numerator (Art. 124). The value of the third

fraction will not be altered if we change the signs of *both* the factors $c-a$ and $c-b$, so as to have $a-c$ and $b-c$. Hence

$$\frac{1}{a(a-b)(a-c)}=\frac{bc(b-c)}{abc(a-b)(a-c)(b-c)};$$

$$\frac{1}{b(b-a)(b-c)}=\frac{-ac(a-c)}{abc(a-b)(a-c)(b-c)};$$

$$\frac{1}{c(c-a)(c-b)}=\frac{ab(a-b)}{abc(a-b)(a-c)(b-c)}.$$

The sum of the numerators is $bc(b-c)-ac(a-c)+ab(a-b)$, which, by performing the multiplications indicated, and reducing gives the same result as $(a-b)(a-c)(b-c)$. Hence, this product may be canceled in both terms, and the sum of the three fractions is found $=\frac{1}{abc}$.

EXAMPLES IN SUBTRACTION OF FRACTIONS.

(14) By reducing the third fraction to its lowest terms it becomes $\frac{x^2}{y(x+y)}$. The second fraction subtracted from the first leaves $\frac{x^2+xy+y^2}{y(x+y)}$. Subtracting the preceding from this leaves $\frac{xy+y^2}{y(x+y)}=1$. *Ans.*

(15) $$\frac{1}{x-1}-\frac{1}{2(x+1)}=\frac{2(x+1)-(x-1)}{2(x^2-1)}=\frac{x+3}{2(x^2-1)};$$

$$\frac{x+3}{2(x^2-1)}-\frac{x+3}{2(x^2+1)}=\frac{(x+3)(x^2+1)-(x+3)(x^2-1)}{2(x^4-1)}=\frac{x+3}{x^4-1}.\quad \textit{Ans}$$

(16) $$\frac{1}{x^3}+\frac{1}{x^2}-\frac{1}{x}=\frac{1+x-x^2}{x^3}.$$

$$\frac{x-1}{x^2+1}-\frac{1}{(x^2+1)^2}=\frac{x^3-x^2+x-1-1}{(x^2+1)^2}=\frac{x^3-x^2+x-2}{(x^2+1)^2}.$$

$$\frac{1+x-x^2}{x^3}=\frac{1+x-x^2}{x^3}\times\frac{(x^2+1)^2}{(x^2+1)^2}=\frac{-x^6+x^5-x^4+2x^3+x^2+x+1}{x^3(x^2+1)^2}$$

$$\frac{x^3-x^2+x-2}{(x^2+1)^2}\times\frac{x^3}{x^3}\qquad=\frac{x^6-x^5+x^4-2x^3}{x^3(x^2+1)^2}$$

$$\text{Sum}\ =\frac{x^2+x+1}{x^3(x^2+1)^2}.\quad \textit{Ans}$$

MULTIPLICATION AND DIVISION OF FRACTIONS.

Article 131.

REMARK.— In the solution of all questions in multiplication or division of fractions, it is important to separate the quantities into factors, before performing any actual multiplications, as this might so involve the factors that they could not be readily discovered. By attention to factoring nearly all the examples are easily solved.

(11)

$$x^2+x+1$$
$$\frac{1}{x^2}-\frac{1}{x}+1$$
$$\overline{1+\frac{1}{x}+\frac{1}{x^2}}$$
$$-x-1-\frac{1}{x}$$
$$+x^2+x+1$$
$$\overline{x^2+1+\frac{1}{x}}$$

(12)

$$\frac{4a}{3x}+\frac{3x}{2b}$$
$$\frac{2b}{3x}+\frac{3x}{4a}$$
$$\overline{\frac{8ab}{9x^2}+1}$$
$$+1+\frac{9x^2}{8ab}$$
$$\overline{\frac{8ab}{9x^2}+2+\frac{9x^2}{8ab}}$$

(14) $pr+(pq+qr)x+q^2x^2=(p+qx)r+(p+qx)(qx=(p+qx)(r+qx)$;
$ps+(pt-qs)x+qtx^2=(p-qx)s+(p-qx)tx=(p-qx)(s+tx)$

The factors in the denominator of the product will cancel the factors $p-qx$ and $p+qx$ in the numerator, leaving for the result $(r+qx)(s+tx)=rs+(rt+qs)x+qtx^2$.

Article 132.

(8) In solving this and the next two examples, first perform the operation indicated in the parentheses. A similar remark applies to example 12.

(14) $x^4-\frac{1}{x^4}=\frac{x^8-1}{x^4}=\frac{(x^4-1)(x^4+1)}{x^4}=\frac{(x^2+1)(x^2-1)(x^4+1)}{x^4}$;

$x-\frac{1}{x}=\frac{x^2-1}{x}$.

$$\frac{x^2+1)(x^2-1)(x^4+1}{x^4}\times\frac{x}{x^2-1}=\frac{(x^2+1)(x^4+1)}{x^3}=\frac{x^6+x^4+x^2+1}{x^3}$$

$$=x^3+x+\frac{1}{x}+\frac{1}{x^3}, \text{ or } x^3+\frac{1}{x^3}+x+\frac{1}{x}.$$

This example may also be readily solved by ordinary division.

Article 133.

REMARK.—In the solution of the examples in this article the first step is to perform the operations indicated in the respective terms. By doing this they are all easily solved, except the 5th, of which the solution is here given.

$$\frac{1}{a}+\frac{1}{ab^3}=\frac{b^3+1}{ab^3}=\frac{(b+1)(b^2-b+1)}{ab^3}, \text{ (Art. 83)};$$

$$b-1+\frac{1}{b}=\frac{b^2-b+1}{b};$$

$$\frac{(b+1)(b^2-b+1)}{ab^3}\times\frac{b}{b^2-b+1}=\frac{b+1}{ab^2}. \quad \textit{Ans.}$$

MISCELLANEOUS EXERCISES IN FRACTIONS.

(2) To reduce these fractions to a common denominator, it will be most convenient to change the signs of the factors as in the solution to Ex. 14, Addition of Fractions, so that the common denominator may be $(a-b)(a-c)(b-c)$.

$$\frac{a^2+a+1}{(a-b)(a-c)}=\frac{a^2+a+1}{(a-b)(a-c)}\times\frac{b-c}{b-c}=\frac{a^2b-a^2c+ab-ac+b-c}{(a-b)(a-c)(b-c)};$$

$$\frac{-b^2-b-1}{(a-b)(b-c)}=\frac{-b^2-b-1}{(a-b)(b-c)}\times\frac{a-c}{a-c}=\frac{-ab^2+b^2c-ab+bc-a+c}{(a-b)(a-c)(b-c)};$$

$$\frac{c^2+c+1}{(a-c)(b-c)}=\frac{c^2+c+1}{(a-c)(b-c)}\times\frac{a-b}{a-b}=\frac{ac^2-bc^2+ac-bc+a-b}{(a-b)(a-c)(b-c)}.$$

$$\text{Sum }=\frac{a^2b-ab^2-a^2c+b^2c+ac^2-bc^2}{a^2b-ab^2-a^2c+b^2c+ac^2-bc^2}=. \quad \textit{Ans.}$$

(3) Perform the operations indicated before substituting the value of x.

(4) $\frac{2x-3}{3}-\frac{3x-1}{4}=-\frac{x+9}{12}$; $-\frac{x+9}{12}\times\frac{2}{x-1}=\frac{-x-9}{6(x-1)}$;

$$\frac{x}{2}-\left\{\frac{-x-9}{6(x-1)}\right\}=\frac{3x(x-1)+x+9}{6\ (x-1)}=\frac{3x^2-2x+9}{6(x-1)}$$

$$=\frac{3\times\frac{169}{9}-8\frac{2}{3}+9}{6\times3\frac{1}{3}}=\frac{56\frac{2}{3}}{20}=\frac{170}{60}=2\tfrac{5}{6}. \quad \textit{Ans.}$$

(6) $x+2a=\frac{4ab}{a+b}+2a=\frac{6ab+2a^2}{a+b}$;

$x-2a=\frac{4ab}{a+b}-2a=\frac{2ab-2a^2}{a+b}$;

1st fraction $=\frac{6ab+2a^2}{2ab-2a^2}=\frac{3b+a}{b-a}=\frac{-a-3b}{a-b}$.

$x+2b=\frac{4ab}{a+b}+2b=\frac{6ab+2b^2}{a+b}$;

$x-2b=\frac{4ab}{a+b}-2b=\frac{2ab-2b^2}{a+b}$;

2nd fraction $=\frac{6ab+2b^2}{2ab-2b^2}=\frac{3a+b}{a-b}$.

$\frac{-a-3b}{a-b}+\frac{3a+b}{a-b}=\frac{2a-2b}{a-b}=2.$ *Ans.*

(7) Substituting the value of x we have

$$\frac{a^n}{2na^n-na^n-nb^n}+\frac{b^n}{2nb^n-na^n-nb^n}$$

$$=\frac{a^n}{na^n-nb^n}+\frac{b^n}{nb^n-na^n}=\frac{a^n}{na^n-nb^n}+\frac{-b^n}{na^n-nb^n}$$

$$=\frac{a^n-b^n}{na^n-nb^n}=\frac{a^n-b^n}{n(a^n-b^n)}=\frac{1}{n}.\quad Ans.$$

(8) $\frac{x+y}{xy}=\frac{\frac{x}{xy}+\frac{y}{xy}}{1}=\frac{1}{y}+\frac{1}{x}$

Similarly $\frac{x-y}{xy}=\frac{\frac{x}{xy}-\frac{y}{xy}}{1}=\frac{1}{y}-\frac{1}{x}$.

{ By dividing both terms of each fraction by xy, and reducing.

(9) Let x represent the value of one fraction, and y that of the other; then $x+y=1$.

Multiply each side of this equality by $x-y$ (see Note, page 61), and we have

$x^2-y^2=x-y$, which proves the proposition.

(10). Let x and y represent the fractions, then

$x-y=\frac{p}{q}$; multiply both sides by q, then

$(x-y)=p$.

Multiply both sides of this equality by $x+y$; then

$$q(x^2-y^2)=p(x+y)$$

or $$p(x+y)=q(x^2-y^2).$$

(11) *First.*— Let $(a-b)(a-c)(b-c)$ be the common denominator of the three fractions, then we must change the signs of the numerator of the second fraction, and the signs of the first factor of the denominator. We must also change the signs of both factors of the denominator of the second fraction. The numerators of the respective fractions when reduced to a common denominator will be

$$(a^2+h^2)(b-c)=\ a^2b-a^2c+bh^2-ch^2;$$
$$(-b^2-h^2)(a-c)=-ab^2+b^2c-ah^2+ch^2;$$
$$(c^2+h^2)(a-b)=\ ac^2-bc^2+ah^2-bh^2.$$

Sum of the numerators $=a^2(b-c)-b^2(a-c)+c^2(a-b)$;

$$(a-b)(a-c)(b-c)=a^2(b-c)-b^2(a-c)+c^2(a-b);$$

hence the value of the fraction is 1.

Second. $$(a^2+h^2)(b-c)(b+c)=\ a^2b^2-a^2c^2+b^2h^2-c^2h^2;$$
$$-(b^2+h^2)(a-c)(a+c)=-a^2b^2+b^2c^2-a^2h^2+c^2h^2;$$
$$(c^2+h^2)(a-b)(a+b)=\ a^2c^2-b^2c^2+a^2h^2-b^2h^2.$$

The sum of the numerators is 0, hence the sum of the fractions is 0.

Third. $$(a^2+h^2)(b-c)bc=\ a^2b^2c-a^2bc^2+b^2ch^2-bc^2h^2;$$
$$-(b^2+h^2)(a-c)ac=-a^2b^2c+ab^2c^2-a^2ch^2+ac^2h^2$$
$$(c^2+h^2)(a-b)ab=\ a^2bc^2-ab^2c^2+a^2bh^2-ab^2h^2.$$

The sum of the numerators is $h^2\{a^2(b-c)-b^2(a-c)+c^2(a-b)\}$ and since the denominator is the quantity within the brackets, the value of the fraction is h^2.

EQUATIONS OF THE FIRST DEGREE.

REMARKS.— The attentive student will find no difficulty with the examples in Articles 151, and 153, provided he attends carefully to the rules. (See Remark on page 14, of the Key to Part First.)

The ease and facility with which several of the examples may be solved, will depend on the particular method of solution The shortest methods, however, are not always the best for learners. It is important even at the risk of being tedious, that the pupil understand every step of the operation. Let the aim be first to perform the operations correctly and understandingly, and after this with facility.

In some cases it is better to perform the operations indicated before clearing the equation of fractions. To illustrate this we will take example 27, Art. 153.

Multiplying the terms in the parentheses in the second member by $\frac{1}{39}$ and removing it, we have

$$\tfrac{1}{2}\left(x-\frac{51}{26}\right)-\frac{2}{13}(1-3x)=x-\frac{5x}{39}+\frac{1-.3x}{15.6}$$

Now 156 is evidently the least common multiple of the denominators. Multiplying both members by this, we have

$$78x-153-24+72x=156x-20x+10-3x;$$

reducing, $17x=187$, whence $x=11$.

QUESTIONS PRODUCING EQUATIONS OF THE FIRST DEGREE.

Article 154.

(9) Let $x=$ the first, then $2x=$ the second, and $3x=$ the third; and $x+2x+3x=133$.
Whence $x=19$, $2x=38$, and $3x=57$.

(10) Let $x=$ the first, then $3x=$ the second, and $4\frac{1}{2}x=$ the third; and $x+3x+4\frac{1}{2}x=187$,
Whence $x=22$, $3x=66$, and $4\frac{1}{2}x=99$.

(11) Let $x=$ the second, then $3\frac{1}{2}x=$ the first, and $3\frac{1}{2}x-x=100$.
Whence $x=40$ and $3\frac{1}{2}x=140$.

(12) Let $x=$ the first, then $3\frac{1}{2}x=$ the second, and $100-(3\frac{1}{2}x-x)=100-2\frac{1}{2}x=$ the third.
Then $x+3\frac{1}{2}x+100-2\frac{1}{2}x=156$.
Whence $x=28$, $3\frac{1}{2}x=98$, and $100-2\frac{1}{2}x=30$.

(13) Let $x=$ the number, then
$\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=52$.
Whence $\frac{13x}{12}=52$, and $x=48$.

(14) Let $x=$ the number, then

$$x+\frac{6x}{7}-20=45.$$

Whence $x=35$.

15) Let $x=$ the number, then

$$x+\frac{x}{3}+\frac{x}{4}-\frac{x}{6}=51.$$

Whence $x=36$.

(16) Let $x=$ the number, then

$4x-40=40-x$.

Whence $x=16$.

17) Let $x=$ the number, then

$4(x+16)=10(x+1)$.

Whence $x=9$.

(18) Let $x=$ the less number, then $30-x=$ the greater,

and $\frac{1}{4}(30-x-x)=3$.

Whence $x=9$, and $30-x=21$.

(19) Let $x=$ the number of days he worked, then $28-x=$ idle days;

then $75x-25(28-x)=1200$.

or $\frac{3}{4}x-\frac{1}{4}(28-x)=12$.

Whence $\frac{3}{4}x-7+\frac{1}{4}x=12$, or $x=19$.

(20) Let $x=$B's money, then $3x=$A's,

and $3x+50=4(x-50)$.

Whence $x=250$, and $3x=750$.

(21 Let $x=$ sum, then $x-\frac{1}{2}x-20=\frac{x}{2}-20$;

$$\frac{x}{2}-20-\frac{1}{3}\left(\frac{x}{2}-20\right)-30=\frac{x}{2}-\frac{x}{6}-50+\frac{20}{3}=\frac{x}{3}-\frac{130}{3}$$

$$\frac{x}{3}-\frac{130}{3}-\frac{1}{4}\left(\frac{x}{3}-\frac{130}{3}\right)-40=0.$$

Whence $x=290$.

(22) Observe that 20 per cent. is $\frac{1}{5}$ and 25 per cent. $\frac{1}{4}$

Let $x=$ capital, then $x+\frac{15x}{100}=\frac{115x}{100}=$ cap. close 1^{st} yr.

$\frac{115x}{100}+\frac{1}{5}$ of $\frac{115x}{100}=\frac{115x}{100}+\frac{23x}{100}=\frac{138x}{100}=$ cap. 2^{nd} yr.

$\frac{138x}{100}+\frac{1}{4}$ of $\frac{138x}{100}=\frac{138x}{100}+\frac{69x}{200}=\frac{345x}{200}=$ cap. 3^{rd} yr.

$\therefore \frac{345x}{200}-x=1000.50.$

Whence $x=1380$.

(23) Let $x=$ B's age, then $2x=$ A's,
and $3(x-22)=2x-22$.
Whence $x=44$, and $2x=88$.

(24) To avoid fractions we may take some multiple of x that is divisible twice by 2. Thus,

Let $4x=$ cost of 1^{st} house, then $4x+2x=6x=$ cost of 2^{nd},
and $6x+3x=9x=$ cost of 3^{rd},
also $4x+9x=13x=$ cost of 4^{th}.
Hence $4x+6x+9x+13x=8000$.
Whence $4x=1000$, $6x=1500$, $9x=2250$, and $13x=3250$.

(25) Let $x=$ gallons third conveys in 1 minute,
then $3x=$ " " " " 3 minutes,
and $3x+8=$ galls first " " 3 "
also $3x-7=$ " second " " 3 "
$\therefore$ $9x+1=$ " all convey " 3 "
$72x+8=$ " " " " 24 "

$\therefore$ $72x+8=1050$.

Whence $x=14\frac{17}{36}$, $\frac{1}{3}(3x+8)=17\frac{5}{36}$; $\frac{1}{3}(3x-7)=12\frac{5}{36}$.

(27) Let $x=$ the number of days in which B can do it,

then $\frac{1}{x}=$ part B does in one day; but A does $\frac{1}{10}$, and A and B together do $\frac{1}{7}$ in one day;

$\therefore \frac{1}{7}-\frac{1}{10}=\frac{1}{x}$. Whence $x=23\frac{1}{3}$.

(28) Let $x=$ the number of days in which A can do it,

then $\frac{1}{x}$= part A does in one day ; but A does $\frac{1}{4}$ of $\frac{2}{7}=\frac{1}{14}$ in one day ; $\therefore$ $\frac{1}{x}=\frac{1}{14}$. Whence $x=14$.

If A and B finish $\frac{5}{7}$ of the work in 6 days, they do $\frac{1}{6}$ of $\frac{5}{7}=\frac{5}{42}$ in one day ; and since A does $\frac{1}{14}$ in one day, B does $\frac{5}{42}-\frac{1}{14}=\frac{1}{21}$ in one day, or the whole in 21 days.

The solution of this question mainly depends on arithmetical analysis, and the employment of algebraic symbols can scarcely be said to be of any advantage.

(29) Let x= number of each, then $3x$= cost of sheep, $12x$= cost of cows, and $18x$= cost of oxen.

$\therefore$ $3x+12x+18x=330$.

Whence $x=10$.

(30) Let x= sum A rec'd, then $x-10$= what B rec'd ; $x-10+16=x+6$= what C rec'd ; $x+6-5=x+1$= what D rec'd ; $x+1+15=x+16$= what E rec'd.

$\therefore$ $x+1+x+16=x+x-10+x+6$.

Whence $x=21$, what A rec'd, from which, what the others rec'd, is readily found.

(31) Let x= the number of eggs, then $\frac{x}{12}$= number of dozen, and $\frac{x}{12}\times 18=\frac{3x}{2}$= cost.

$\frac{x+5}{12}$= number of dozen if he had bought 5 more, and since the whole cost divided by the number of dozen, must give the cost of one dozen, therefore

$\frac{3x}{2}\div\frac{x+5}{12}$= cost of one dozen under second supposition

$\frac{3x}{2}\div\frac{x+5}{12}=\frac{3x}{2}\times\frac{12}{x+5}=\frac{18x}{x+5}$;

$\therefore$ $\frac{18x}{x+5}=18-2\frac{1}{2}=15\frac{1}{2}$.

Whence $x=31$.

(32) Let x= the number bought, then $\frac{94}{x}$= cost of each and $\frac{1}{4}(x-7)$= one-fourth of the remainder.

$20\div\frac{1}{4}(x-7)=20\times\frac{4}{x-7}=\frac{80}{x-7}=$ what each sold for.

$\therefore\ \frac{94}{x}=\frac{80}{x-7}$. Whence $x=47$.

(33) Let $x=$ the number of hours each traveled, then $\frac{x}{2}\times 3$ $=\frac{3x}{2}=$ miles A traveled, and $\frac{x}{4}\times 5=\frac{5x}{4}=$ miles B traveled;

$\therefore\ \frac{3x}{2}+\frac{5x}{4}=154$.

Whence $x=56$, $\frac{3x}{2}=84$, and $\frac{5x}{4}=70$.

(34) Let $x=$ the number, then

$\frac{5x-24}{6}+13=x$.

Whence $x=54$.

(35) Let $x=$ number of dollars, then $3x=$ number of eagles.

$\therefore\ 5(x-8)=3x-8$.

Whence $x=16$, and $3x=48$.

(36) Let $x=$ number of apples, then $100-x=$ number of pears

then $\frac{x}{10}\times 1=\frac{x}{10}=$ cost of apples;

and $\frac{100-x}{25}\times 2=\frac{200-2x}{25}=$ cost of pears;

$\therefore\ \frac{x}{10}+\frac{200-2x}{25}=9\frac{1}{2}$.

Whence $x=75$, and $100-x=25$

(37) Let $x=$ number of sheep,

then $\frac{x}{8}=$ acres ploughed, and $\frac{x}{5}=$ acres of pasture;

$\therefore\ \frac{x}{8}+\frac{x}{5}=325$. Whence $x=1000$.

(38) Let $x=$ miles he can ride,

then $\frac{x}{12}=$ time of riding and $\frac{x}{4}=$ time of walking;

$\therefore\ \frac{x}{12}+\frac{x}{4}=2$. Whence $x=6$.

(39) Let $x=$ number of lbs, then $\frac{2}{65+x}=$ lbs of salt in 1 lb,

and $25\left(\frac{2}{65+x}\right)=$ lbs of salt in 25 lbs.

$\therefore\ 25\left(\frac{2}{65+x}\right)=\frac{1}{4}$. Whence $x=135$.

(40) In every 10 lbs of the mass there are 7 lbs of copper and 3 lbs of tin; hence in 80 lbs there are $\frac{80}{10}\times7=56$ lbs of copper, and $\frac{80}{10}\times3=24$ lbs of tin.

Let $x=$ lbs of copper to be added, then $56+x=$ lbs of copper in the new mass, and $24=$ lbs of tin; and since there are 11 lbs of copper for every 4 lbs of tin, one-eleventh of the copper must be equal to one-fourth of the tin.

$\therefore\ \frac{56+x}{11}=\frac{24}{4}$. Whence $x=10$.

(41) Let $x=$ stock, then $x-250+\frac{1}{3}(x-250)=\frac{4x}{3}-\frac{1000}{3}=$ stock at the close of the 1st year.

$\frac{4x}{3}-\frac{1000}{3}-250=\frac{4x}{3}-\frac{1750}{3}$; and $\frac{4x}{3}-\frac{1750}{3}+\frac{1}{3}\left(\frac{4x}{3}-\frac{1750}{3}\right)$

$=\frac{16x}{9}-\frac{7000}{9}=$ stock at close of 2nd year.

$\frac{16x}{9}-\frac{7000}{9}-250=\frac{16x}{9}-\frac{9250}{9}$; and $\frac{16x}{9}-\frac{9250}{9}$

$+\frac{1}{3}\left(\frac{16x}{9}-\frac{9250}{9}\right)=\frac{64x}{27}-\frac{37000}{27}=$ stock at close of 3r yr

$\therefore\ \frac{64x}{27}-\frac{37000}{27}=2x$.

Whence $x=3700$

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE, CONTAINING TWO UNKNOWN QUANTITIES

Article 158.

(17) Divide the second equation by $a-b$ and we have

$(a+b)(x+y)=\frac{n}{a-b}$;

or $(a+b)x+(a+b)y=\frac{n}{a-b}$; (3).

$(a-b)x+(a+b)y=c$; equation (1)

$2bx=\frac{n}{a-b}-c$; by subtracting (1) from (3)

$\therefore\ x=\frac{1}{2b}\left(\frac{n}{a-b}-c\right)$.

Again, dividing the second equation by $a+b$, we have

$(a-b)(x+y)=\frac{n}{a+b}$,

or $(a-b)x+(a-b)y=\frac{n}{a+b}$; (4)

$(a-b)x+(a+b)y=c$; (1)

$2by=c-\frac{n}{a+b}$, by subtracting (4) from (1).

$\therefore\ y=\frac{1}{2b}\left(c-\frac{n}{a+b}\right)$.

(18). By multiplying eq. (1) by m, and (2) by n, and subtracting, we find the value of x. Again, by multiplying (1) by n and (2) by m, and subtracting, we find the value of y.

(19) Multiplying both equations by abc, transposing and factoring, we have

$(bc+ab)x+acy=abc$; (1)

$acx+(bc-ab)y=abc$; (2)

Multiplying the first equation by $bc-ab$, and the second by ac, we have

$(b^2c^2-a^2b^2)x+(bc-ab)acy=abc(bc-ab)$; (3)

$a^2c^2x+(bc-ab)acy=a^2bc^2$. (4)

Subtracting equation (3) from (4), and factoring, we have

$(a^2b^2+a^2c^2-b^2c^2)x=abc(ab+ac-bc)$.

Whence $x=\frac{abc(ab+ac-bc)}{a^2b^2+a^2c^2-b^2c^2}$.

Similarly, we may find the value of y by multiplying equation (1) by ac, and (2) by $bc+ab$, and subtracting.

(20) Transposing b^2y in equation (2), multiplying by 3, and factoring, we have

$(a^2-b^2)3y+(a+b+c)3bx=(a+2b)3ab+\frac{3ab^2c}{a+b}$; (4)

Separating equation (1) into its parts, we have

$(a^2-b^2)5x+(a^2-b^2)3y=(4a-b)2ab$; (1)

Subtracting equation (4) from (1) we have

$$(5a^2-5b^2-3ab-3b^2-3bc)x=8a^2b-2ab^2-3a^2b-6ab^2-\frac{3ab^2c}{a+b}$$

Reducing and factoring,

$$(5a^2-8b^2-3ab-3bc)x=\frac{ab}{a+b}(5a^2-8b^2-3ab-3bc).$$

Whence $x=\frac{ab}{a+b}$.

Substituting the value of x in equation (1) we have

$\frac{5ab}{a+b}(a^2-b^2)+3y(a^2-b^2)=8a^2b-2ab^2$; reducing,

$3y(a^2-b^2)=3a^2b+3ab^2=3ab(a+b)$,

or, $y(a-b)=ab$;

$\therefore\ y=\frac{ab}{a-b}$.

QUESTIONS PRODUCING SIMULTANEOUS EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

Article 159.

(4) Let $x=$ number of sheep, and $y=$ number of cows then $5x+7y=111$,

$7x+5y=93$.

Whence $x=4$ and $y=13$.

(5) Let $x=$ cost of 1 lb tea, and $y=$ cost of 1 lb [illegible] then $7x+9y=520$,

$4x+11y=385$.

Whence $x=55$, and $y=15$ cts.

(6) Let $x=$ A's money, and $y=$ B's, then $x+50=y-20$,

$3x+5y=2350$.

Whence $x=250$, and $y=320$.

(7) Let $6x=$ A's money, and $5y=$ B's, then $6x+5y=9800$

also $6x-x=5y-y$, or $5x-4y=0$.
Whence $x=800$ and $y=1000$;
$\therefore$ $6x=4800$ and $5y=5000$.

(8) Let $x=$ the numerator and y the denominator of the fraction,
then $\frac{x+1}{y+1}=\frac{1}{2}$, and $\frac{x-1}{y-1}=\frac{1}{3}$.
Whence $x=3$, and $y=7$.

(9) Let $x=$ the first number and $y=$ the second,
then $\frac{x}{3}=\frac{y}{4}+3$,
$\frac{x}{4}+\frac{y}{5}=10$.
Whence $x=24$ and $y=20$.

(10) Let $x=$ number of lbs, and $y=$ cost per lb, then $xy=$ cost
$\therefore$ $30x-xy=100$, (1)
$xy-22x=300$. (2)
Adding equations (1) and (2) together we have
$8x=400$, whence $x=50$.
By substitution, the value of y is found $=28$.

(11) Let $x=$ number of bushels of wheat, and $y=$ bushels of corn,
then $55x=33y$; and $55x+33y=$ rent;
also $65x+41y-140=$ rent.
$\therefore$ $65x+41y-140=55x+33y$,
or $10x+8y=140$.
Whence $x=6$ and $y=10$.

(12) Let x and $y=$ the cubic feet which each discharges,
then $x : y :: 5\times8 : 13\times7$;
$\therefore$ $40y=91x$; (1)
also $y-x=561$. (2)
Whence $x=440$ and $y=1001$.
From (1) it is evident that y is greater than x, therefore in (2) we write $y-x$.

(13) Let $5x$ and $7x$ represent the first two numbers, and $3y$ and $5y$ the other two, then

$5x+3y : 7x+5y :: 9 : 13$;
$\therefore\ 65x+39y=63x+45y$, or $2x=6y$; (1)
also $7x+5y-(5x+3y)=16$,
or $2x+2y=16$. (2)
Whence $x=6$ and $y=2$, $\therefore\ 5x=30$, $7x=42$; and $3y=6$ and $5y=10$.

(14) Let $x=$ number of apples, and $y=$ number of pears,
then $\frac{x}{4}+\frac{y}{5}=30$,
and $\frac{1}{2}$ of $\frac{x}{4}+\frac{1}{3}$ of $\frac{y}{5}$, or $\frac{x}{8}+\frac{y}{15}=13$.
Whence $x=72$ and $y=60$.

(15) Let $x=$ acres of tillable land, and $y=$ acres of pasture,
then $200x+140y=24500$; (1)
also $x : \frac{x-y}{2} :: 28 : 9$;
$\therefore\ 9x=14x-14y$, or $5x=14y$. (2)
Whence $x=98$, and $y=35$.

(16) Let $x=$ digit in ten's place, and $y=$ digit in unit's place, then $10x+y=$ the number, and $10y+x=$ the number when the digits are inverted,
then $10x+y+10y+x=121$,
or $11x+11y=121$; (1)
and $10x+y-(10y+x)=9$,
or $9x-9y=9$. (2)

Dividing (1) by 11, and (2) by 9, and adding and subtracting we find $x=6$, and $y=5$.

REMARK.— It may be asked why, in obtaining equation (2), do we subtract $10y+x$ instead of $10x+y$, since we do not know which is the greater. The answer is, we can not tell which to subtract till we proceed to verify the result, but if we had subtracted the wrong quantity, the error would be made known in verifying the result, by some quantity being negative that ought to be positive. (See Art. 164.)

(17) Let $2x-6$, $3x-6$, and y be the numbers, which fulfils the first condition. The second condition gives
$2x-1 : y+5 :: 7 : 11$,
or $22x-11=7y+35$; (1)
also $3x-42 : y-36 :: 6 : 7$,

or $21x-294=6y-216$. (2)
Whence $x=18$, and $y=50$;
$\therefore$ the numbers are 30, 48, and 50

(18) Let x and z represent the days respectively in which A and B can do it,

then $\frac{1}{x}$ and $\frac{1}{z}=$ parts which each can do in a day.

Then $\frac{1}{x}+\frac{1}{z}=\frac{1}{16}$. (1)

Also in 4 days A and B do $\frac{4}{x}+\frac{4}{z}$, and in 36 days B does $\frac{36}{z}$ parts of the work;

$\therefore \frac{4}{x}+\frac{4}{z}+\frac{36}{z}=1$ (the whole work);

or $\frac{4}{x}+\frac{40}{z}=1$. (2)

Multiplying equation (1) by 4, and subtracting it from (2) we have $\frac{36}{z}=\frac{3}{4}$; whence $z=48$, and by substitution is readily found $=24$.

(19) First, 2 hrs 48 min. $=2\frac{4}{5}$ hrs, and 4 hrs, 40 min. $=4\frac{2}{3}$ hrs.

Let x and z represent the hours respectively in which A and B can drink it, then $\frac{1}{x}$ and $\frac{1}{z}=$ parts which each can drink in an hour; and $\frac{2}{x}+\frac{2}{z}=$ parts drank by both in 2 hours

$\frac{2\frac{4}{5}}{z}=\frac{14}{5z}=$ parts drank by B in $2\frac{4}{5}$ hours;

$\frac{4\frac{2}{3}}{x}=\frac{14}{3x}=$ parts drank by A in $4\frac{2}{3}$ hours;

$\therefore \frac{2}{x}+\frac{2}{z}+\frac{14}{5z}=1$ (the whole); (1)

$\frac{2}{x}+\frac{14}{3x}+\frac{2}{z}=1$ " (2).

By adding together the terms containing x, and those containing z,

$\frac{2}{x}+\frac{24}{5z}=1$; (3)

$\frac{20}{3x}+\frac{2}{z}=1$. (4)

By multiplying (3) by $\frac{10}{3}$ and subtracting (4) from it, we have $\frac{14}{z}=\frac{7}{3}$ whence $z=6$, and by substitution x is easily found $=10$.

(20) Let $x=$ numerator and $y=$ denominator of 1st fraction, then $\frac{x}{y}=$ 1st fraction, and $\frac{8}{5}-\frac{x}{y}=\frac{8y-5x}{5y}=$ 2nd fraction.

By adding the numerators together, and the denominators together, we have

$x+8y-5x=y+5y$,

or $2y=4x$, or $2x=y$;

whence $\frac{x}{y}=\frac{1}{2}=$ the first fraction,

and $\frac{8}{5}-\frac{1}{2}=\frac{11}{10}=$ the second fraction.

(21) In solving questions of this kind, it is convenient to denote the capacity by 1; it may, however, be denoted by c, the object of the question being not to find either the size of a crown or guinea, or the size (capacity) of the purse, but the *ratio* of the size of a crown or guinea to the size of the purse.

Let $x=$ number of crowns and $z=$ number of guineas, then $\frac{1}{x}=$ part filled by 1 crown, and $\frac{1}{z}=$ part filled by 1 guinea.

Also, $\frac{19}{x}+\frac{6}{z}=1$, (1)

and $\frac{4}{x}+\frac{5}{z}=\frac{17}{63}$. (2)

Multiplying equation (1) by 5 and (2) by 6, and subtracting, we find $x=21$; then by substitution $z=63$.

22) Let $x=$ number of bushels of wheat, and $y=$ number of bushels of rye.

$\therefore 5x+3y=$ his money.

Observe that 7 bushels of rye will cost 21 shillings, and 6 bushels of wheat 30 shillings. Then from the nature of the question, we have the following equations·

$$\frac{5x+3y-21}{5}+7=x+y-2, \qquad (1)$$

$$30+3(x+y-6)=5x+3y-6. \qquad (2)$$

Whence $x=9$ and $y=12$.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE, INVOLVING THREE OR MORE UNKNOWN QUANTITIES.

Article 160.

(9) (3) from (2) gives $\frac{1}{x}-\frac{1}{y}=b-c$; (4)

Sum of (1) and (4) gives $\frac{2}{x}=a+b-c$.

Whence $x=\frac{2}{a+b-c}$.

(2) from (1) gives $\frac{1}{y}-\frac{1}{z}=a-b$; (5)

Sum of (3) and (5) gives $\frac{2}{y}=a-b+c$.

Whence $y=\frac{2}{a-b+c}$.

(1) from (3) gives $\frac{1}{z}-\frac{1}{x}=c-a$, (6)

Sum of (2) and (6) gives $\frac{2}{z}=b+c-a$.

Whence $z=\frac{2}{b+c-a}$.

(10) Multiplying equation (1) by 3, and (2) by 2, to render tne coëfficients of $\frac{1}{x}$ alike, and subtracting the former from the latter, we have

$$-\frac{17}{y}+\frac{22}{z}=\frac{4}{3}. \qquad (4)$$

Multiplying equation (1) by 2 and adding the result to (3) we have $\frac{11}{y}-\frac{2}{z}=\frac{2}{3}$. (5)

Multiplying equation (5) by 11, and adding the result to (4), we have $\frac{104}{y}=\frac{26}{3}$; whence $y=12$

(11) Multiplying equation (1) by 2, and subtracting (2) from the result, we have $\frac{15}{4x}-\frac{13}{3y}=\frac{31}{216}$. (4)

Multiplying equation (2) by 2, and subtracting (3) from the result, we find $-\frac{1}{3x}+\frac{3}{y}=\frac{5}{18}$. (5)

Multiplying equation (4) by 3, and (5) by $\frac{13}{3}$, and adding and reducing, we find $x=6$; then by going back and substituting, we readily find the values of y and z.

(12) Adding the four equations together, and dividing by 2, we find the value of $x+y+z+v$. Then subtracting from this each of the equations successively, and dividing by 2, we get the values of x, y, z, and v.

QUESTIONS PRODUCING SIMULTANEOUS EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

Article 161.

(1) Let x, y, and z represent the respective shares, then

$$x+y+z=760, \quad (1)$$
$$x+y-z=240, \quad (2)$$
$$y+z-x=360. \quad (3)$$

Whence $x=200$, $y=300$, and $z=260$.

(2) Let x, y, and z represent the numbers respectively, then

$$x+y+z=20; \quad (1)$$
$$x+y : y+z :: 4 : 5, \text{ or } 5x+5y=4y+4z; \quad (2)$$
$$y-x : z-x :: 2 : 3, \text{ or } 3y-3x=2z-2x. \quad (3)$$

Whence $x=5$, $y=7$, and $z=8$.

(3) Let x, y, z, and v represent the numbers respectively, then

$$x+y+z=13, \quad (1)$$
$$x+y+v=15, \quad (2)$$
$$x+z+v=18, \quad (3)$$
$$y+z+v=20. \quad (4)$$

Adding the four equations together and dividing by 3, we have $x+y+z+v=22$, from which, by subtracting equations (4), (3), (2), and (1) respectively, we find $x=2$, $y=4$, $z=7$, and $v=9$.

(4) Let $x=$ digit in hundred's place, $y=$ digit in ten's place, and $z=$ digit in unit's place, then $100x+10y+z=$ the number, and $x+y+z=16$; (1)
also $x+y : y+z :: 3 : 3\frac{2}{3}$, or $3\frac{2}{3}x+3\frac{2}{3}y=3y+3z$; (2)
and $100x+10y+z+198=100z+10y+x$;
or $99x+198=99z$. (3)
From these equations we readily find $x=5$, $y=4$, and $z=7$.

(5) Let $x=$ number of votes for A and B, $y=$ do. for A and C, and $z=$ do. for B and C; then

$$x+y+26=158, \quad (1)$$
$$x+z+30=132, \quad (2)$$
$$y+z+28=58. \quad (3)$$

Whence $x=102$, $y=30$, and $z=0$.

(6) If x, y, and z represent the three numbers, then

$$\tfrac{1}{2}x+\tfrac{1}{3}y+\tfrac{1}{4}z=46, \quad (1)$$
$$\tfrac{1}{3}x+\tfrac{1}{4}y+\tfrac{1}{5}z=35, \quad (2)$$
$$\tfrac{1}{4}x+\tfrac{1}{5}y+\tfrac{1}{6}z=28\tfrac{1}{3}. \quad (3)$$

By clearing these equations of fractions, the values of x, y, and z are readily found by elimination by addition and subtraction.

(7) Let x, y, and z represent the three numbers, then

$$x+y=a, \quad (1)$$
$$x+z=b, \quad (2)$$
$$y+z=c. \quad (3)$$

Whence x, y, and z are readily found.

(8) Let x, y, z, and v represent the capacity of the respective casks, then

$$x-y=\frac{4x}{7}, \quad (1)$$

$$y-z=\frac{y}{4}, \quad (2)$$

$z=\frac{9v}{16}$, (3)

$x=z+v+15.$ (4)

Whence $x=140$, $y=60$, $z=45$, and $v=80$.

(9) Let x, y, and z, represent the number of guns, soldiers and sailors, respectively, then

$\frac{x}{3}\times 22+10=z$, (1)

$y+z=5(x+y)$, (2)

Since the number slain in the engagement was one-fourth of the survivors; therefore, $\frac{1}{5}(y+z)$ represents the slain, and $\frac{4}{5}(y+z)$ the survivors.

$\therefore\ \frac{4}{5}(y+z)+5=\frac{x}{2}\times 13.$ (3)

From these equations we readily find $x=90$, $y=55$ and $z=670$.

GENERALIZATION.

Article 163.

(8) Representing the parts by $x-m$, $x+m$, $\frac{x}{m}$, and mx, we have

$x-m+x+m+\frac{x}{m}+mx=a$;

$$2x+\frac{x}{m}+mx=a;$$

$2mx+x+m^2x=ma$;

$x(m^2+2m+1)=x(m+1)^2=ma.$

Whence $x=\frac{ma}{(m+1)^2}$, from which the parts are easily found.

(9) Let $x=$ distance he may ride, then

$\frac{x}{b}=$ time employed in riding, and

$\frac{x}{c}=$ time " " walking.

$\therefore\ \frac{x}{b}+\frac{x}{c}=a$; whence $x=\frac{abc}{b+c}$.

(10) Let $x=$ the less number, then $bx=$ the greater, since the quotient of the greater divided by the less is b.

$\therefore\ bx+x=a$, or $(b+1)=a$.

Whence $x=\frac{a}{b+1}=$ less, and $bx=\frac{ab}{b+1}=$ greater.

(11) Let $x=$ the number of beggars that received b cts. each, then $n-x=$ the number that received c cts each.

$\therefore\ bx+c(n-x)=a$.

Whence $x=\frac{a-nc}{b-c}$, and $n-x=\frac{nb-a}{b-c}$.

(12) Let $x=$ the greater part, and $n-x=$ the less, then

$\frac{x}{n-x}=q+\frac{r}{n-x}$.

Whence $x=\frac{nq+r}{1+q}$, and $n-x=\frac{n-r}{1+q}$.

(13) Let x, y, and z represent the days respectively in which A, B, and C can perform the work.

Then, if A can do it in x days, he can do $\frac{1}{x}$ part in one day; in like manner B can do $\frac{1}{y}$ part, and C $\frac{1}{z}$ part in one day.

$$\therefore\ \frac{1}{x}+\frac{1}{y}=\frac{1}{a}; \qquad (1)$$

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{b}; \qquad (2)$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{c}. \qquad (3)$$

For the method of solution see example 9, Art. 160.

(14) Let $x=$ A's share, then $\frac{x}{a}=$ expense of one ox for m months, and $\frac{x}{a}\div m=\frac{x}{ma}=$ expense of one ox for 1 month

$\therefore\ \frac{x}{ma}\times nb=\frac{nbx}{ma}=$ B's share, and

$\frac{x}{ma}\times pc=\frac{pcx}{ma}=$ C's share,

$\therefore\ x+\frac{nbx}{ma}+\frac{pcx}{ma}=P.$

Whence $x=\frac{maP}{ma+nb+pc}$, from which the shares of B and C are easily found.

(15) Let $x=$ cost of 1 lb of the mixture, then $(a+b+c)x=$ cost of the whole mixture.

But $ma=$ cost of a lbs at m shillings per lb,

$nb=$ " b " n " "

$pc=$ " c " c " "

$\therefore\ (a+b+c)x=ma+nb+pc.$

Whence $x=\frac{ma+nb+pc}{a+b+c}$.

(16 Instead of representing either of the quantities to be found by a separate symbol, the simplest solution is obtained by taking x to represent the number of miles per hour the waterman goes when he rows *with* the current; then since he can row c miles with the current for d miles against it, we have

$$c : d : : x : \frac{dx}{c}= \text{rate of sailing up stream.}$$

And since the number of hours employed in sailing any given distance, is equal to the whole number of miles sailed, divided by the number of miles sailed in 1 hour therefore,

$\frac{a}{x}=$ number of hours in sailing down stream, and

$a\div\frac{dx}{c}=\frac{ac}{dx}=$ number of hours in sailing up stream.

$\therefore\ \frac{a}{x}+\frac{ac}{dx}=b$, whence $x=\frac{ac+ad}{bd}$,

and $\frac{dx}{c}=\frac{ac+ad}{bc}$.

$a\div\frac{ac+ad}{bd}=\frac{bd}{c+d}=$ time down;

$a\div\frac{ac+ad}{bc}=\frac{bc}{c+d}=$ time up.

It is evident that the rate of the current will be half the difference of the rates of sailing down and up; that is

$$\frac{1}{2}\left\{\frac{ac+ad}{bd}-\frac{ac+ad}{bc}\right\}=\frac{a(c^2-d^2)}{2bcd}.$$

Lastly, the rate of rowing will be the difference between the rate of sailing and the rate of the current; that is,

$$\frac{ac+ad}{bd}-\frac{a(c^2-d^2)}{2bcd}=\frac{a(c+d)^2}{2bcd}.$$

NEGATIVE SOLUTIONS.

Article 164.

Enunciations of questions 2, 3, 4, 5, and 6, so that the results shall be true in an arithmetical sense.

2. What number must be *subtracted* from the number 30, that the *remainder* shall be 19? *Ans.* 11.

3. The *difference* of two numbers is 9, and their *sum* 25; required the numbers. *Ans.* 17 and 8.

4. What number is that whose *third* subtracted from its *half* leaves a remainder 15? *Ans.* 90.

5. A father's age is 40 years; his son's age is 13 years; *how many years since* was the age of the father 4 times that of the son? *Ans.* 4.

6. The triple of a certain number *increased* by 100, is equal to 4 times the number *diminished* by 200. Required the number. *Ans.* 300.

Article 169.

(1) Here we find $x=\frac{mnq}{m-n}$.

1st. There will be a negative solution when n is greater than m. 2nd. The value of x will be infinite when m is equal to n (Art. 136). 3rd. When q is 0, and m is equal to n, there will be an indeterminate solution; that is, x may have any value whatever.

(2) 1st. The boats will meet half way between C and L, when m is equal to n. 2nd. They will meet at C when m is 0. 3rd. They will meet at L when n is 0. 4th. They

will meet above C when m is less than n, and the boat A runs in an opposite direction from C to L. 5^{th}. They will meet below L when m is greater than n, and the boat B runs in an opposite direction from L to C. 6^{th}. They will never meet if m and n have different signs and are equal to each other. 7^{th}. They will sail together when a is zero, and $-m=n$, or $m=-n$.

(3) Let $x=$ the number, the

$$\frac{8x+16}{4}=2x-7;$$

whence $2x+4=2x-7$, or $11=0$.

This result is *absurd*, therefore the question is absurd or impossible.

(4) Let $x=$ A's age, then $x-6=$ B's, and $x-10=$ C's;

$$\therefore \frac{x}{3}+\frac{x-10}{4}=\frac{7}{12}(x-6)+1.$$

or $4x+3x-30=7x-42+12.$

$$0=0.$$

Hence x may have *any value whatever*, thus if A is 30 years of age, B will be 24, and C 20.

(5) We shall find the same values for x and y from any two of the equations, for example, from the 1^{st} and 2^{nd}, 1 and 3^{rd}, 1^{st} and 4^{th}, 2^{nd} and 3^{rd}, 2^{nd} and 4^{th}, or 3^{rd} and 4^{th}. Hence we may take either two of the equations and the other two will be redundant.

(6) From the 1^{st} and 2^{nd} equations we readily find $x=5$ and $y=3$. From the 1^{st} and 3^{rd}, $x=6\frac{3}{7}$, and $y=2\frac{2}{7}$. From the 1^{st} and 4^{th} $x=-5$ and $y=8$. Hence the equations can not *all* be true at the same time.

EXAMPLES INVOLVING THE SECOND POWER OF THE UNKNOWN QUANTITY.

Article 171.

(9) First divide both members by x^m.

(11) Let $x=$ the number, then

$$4\left(\frac{x}{2}\times\frac{x}{2}\right)=\frac{x}{3}\times\frac{x}{3}\times\frac{x}{3}, \text{ or } x^2=\frac{x^3}{27}.$$

Multiplying both members by 27 and dividing by x^2, we find $x=27$.

(12) Let $x=$ the length and $y=$ the breadth, then $xy=$ the number of square feet. From 1st supposition

$(x+4)(y+5)=xy+116.$ (1)

From the second supposition

$(x+5)(y+4)=xy+113.$ (2)

Performing the operations indicated, omitting xy on each side, and reducing, we have

$5x+4y=96$;

$4x+5y=93$.

Whence $x=12$ and $y=9$.

INVOLUTION OR THE FORMATION OF POWERS.

NOTE.—Most of the examples in the Formation of powers, and the Extraction of roots, being performed by direct methods of operation, which the attentive student will readily understand, it is not deemed necessary to give these solutions here. Those only will be given which present some peculiarity.

Article 172.

There is a theorem by means of which the cube of any binomial may be written directly, which the pupil will sometimes find useful, viz.:

THEOREM.—*The cube of any binomial is equal to the sum of the cubes of the two terms, plus three times their product multiplied by the binomial, if the second term is positive, or minus three times their product multiplied by the binomial, if the second term is negative.* Thus,

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3=a^3+b^3+3ab(a+b);$$

$$(a-b)^3=a^3-3a^2b+3ab^2-b^3=a^3+(-b)^3-3ab(a-b);$$

which proves the theorem.

This theorem gives at once the results in examples 25, 27, and 28. Observe that $x\times\frac{1}{x}=1$, $e^x\times c^{-x}=e^x\times\frac{1}{e^x}=1$.

(29) Let a and $a+1$ be two consecutive numbers

then $(a+1)^2=a^2+2a+1$:

$(a)^2=a^2$

diff. $=2a+1=a+(a+1)$.

(30) Let $a-1$, a, and $a+1$, be any three consecutive numbers; then $(a-1)+a+(a+1)=3a=$ their sum.

$(a-1)^3= a^3-3a^2+3a-1$,

$(a)^3 \quad = a^3$,

$(a+1)^3= a^3+3a^2+3a+1$,

Sum $=3a^3 \quad +6a=3a(a^2+2)$ which is evidently divisible by $3a$.

The theorem may be proved in a similar manner by assuming a, $a+1$, and $a+2$ for the numbers.

EXTRACTION OF THE SQUARE ROOT OF POLYNOMIALS.

Article 183.

(13) The terms arranged with reference to x, give

$$49x^4-\frac{14x^3}{5}+\frac{1051x^2}{25}-\frac{6x}{5}+9.$$

(17)

$$1-x^2 \quad \Big| \; 1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}-, \text{\&c.}$$

$$1$$

$$2-\frac{x^2}{2} \;\Big|\; -x^2$$

$$-x^2+\frac{x^4}{4}$$

$$2-x^2-\frac{x^4}{8} \;\Big|\; -\frac{x^4}{4}$$

$$-\frac{x^4}{4}+\frac{x}{8}+\frac{x^8}{64}$$

$$2-x^2-\frac{x^4}{4}-\frac{x^6}{16} \;\Big|\; -\frac{x^6}{8}-\frac{x^8}{64}$$

$$-\frac{x^6}{8}+\frac{x^8}{16}+\frac{x^{10}}{64}+\frac{x^{12}}{256}$$

$$-\frac{5x^8}{64}-\frac{x^{10}}{64}-\frac{x^{12}}{256}.$$

A more elegant method of extracting the square root of $1-x^2$, is by means of Indeterminate coëfficients, Art. 317; or, by the Binomial theorem, Art. 321.

(18) The operations in this example are similar to those in the preceding.

EXTRACTION OF THE CUBE ROOT OF POLYNOMIALS.

Article 191.

(6) In solving this example let $a+1$ be considered a single quantity. It may, for example, be represented by a single letter as b.

(7

$$1-x \quad \Big| \; 1-\frac{x}{3}-\frac{x^2}{9}-, \text{\&c.}$$
$$1$$
$$3-x+\frac{x^2}{9} \;\Big|\; -x$$
$$-x+\frac{x^2}{3}-\frac{x^3}{27}$$
$$3 \;\Big|\; -\frac{x^2}{3}+\frac{x^3}{27}$$

EXTRACTION OF THE FOURTH ROOT, SIXTH ROOT, &C.

Article 192.

$$\Big|\; x^4-2x^2+3-\frac{2}{x^2}+\frac{1}{x^4}= \text{sq. root.}$$

(11

$$x^8-4x^6+10x^4-16x^2+19-\frac{16}{x^2}+\frac{10}{x^4}-\frac{4}{x^6}+\frac{1}{x^8}$$
$$x^8$$
$$2x^4-2x^2 \;\Big|\; -4x^6+10x^4$$
$$-4x^6+4x^4$$
$$2x^4-4x^2+3 \;\Big|\; 6x^4-16x^2+19$$
$$6x^4-12x^2+9$$
$$2x^4-4x^2+6-\frac{2}{x^2} \;\Big|\; -4x^2+10-\frac{16}{x^2}+\frac{10}{x^4}$$
$$-4x^2+8-\frac{12}{x^2}+\frac{4}{x^4}$$
$$2x^4-4x^2+6-\frac{4}{x^2}+\frac{1}{x^4} \;\Big|\; +2-\frac{4}{x^2}+\frac{6}{x^4}-\frac{4}{x^6}+\frac{1}{x^8}$$
$$+2-\frac{4}{x^2}+\frac{6}{x^4}-\frac{4}{x^6}+\frac{1}{x^8}$$

The square root of $x^4-2x^2+3-\frac{2}{x^2}+\frac{1}{x^4}$ is now readily found to be $x^2-1+\frac{1}{x^2}$.

(12) The terms arranged with reference to the powers of a, give $a^6-6a^4+15a^2-20+\frac{15}{a^2}-\frac{6}{a^4}+\frac{1}{a^8}$. The square root of this, found as in the preceding example, is $a^3-3a+\frac{3}{a}-\frac{1}{a^3}$; and the cube root of this, found by the rule in Art. 191, is $a-\frac{1}{a}$.

It is proper to remark that both the preceding examples may be solved without using fractions in the operation, by multiplying all the terms of the polynomial in example 11, by x^8, and writing x^8 beneath it, and after extracting the fourth root of both terms, dividing by x^2. We should thus find $x^8-2x^6+3x^4-2x^2+1$ for the first square root of the numerator, and x^4-x^2+1 for the second. Similarly, in example 12, we must multiply all the terms by a^6. It is recommended to the pupil to solve these examples by both methods.

RADICALS.

NOTE.— As most of the examples in Radicals are performed by direct methods of operation which the careful student can scarcely fail to apply properly, it is not deemed necessary to present all their solutions.

REDUCTION OF RADICALS.

In the reduction of fractional radicals of the second degree there is a principle with which it is well pupils should be acquainted, as it both facilitates and simplifies the operations. This principle is, that *if a number contains a factor that is a perfect square, the number may be made a perfect square by multiplying it by the other factor*. Thus, if the denominator of a fraction is a^2b it may be made a square by multiplying it by b. For example,

$$\sqrt{\frac{5}{72}}=\sqrt{\frac{5}{36\times2}}=\sqrt{\frac{5\times2}{36\times2^2}}=\sqrt{\frac{1}{36\times4}\times10}=\frac{1}{6\times2}\sqrt{10}=\frac{1}{12}\sqrt{10}.$$

If the denominator contains no factor that is a perfect square, it can only be rendered a perfect square by multiplying both terms by itself. Thus,

$$\sqrt{\tfrac{5}{11}}=\sqrt{\tfrac{5}{11}\times\tfrac{11}{11}}=\sqrt{\tfrac{1}{121}\times 55}=\tfrac{1}{11}\sqrt{55}.$$

Article 199.

(3) $\sqrt{(x^2-y^2)(x+y)}=\sqrt{(x+y)(x-y)(x+y)}=(x+y)\sqrt{x-y}.$

Article 200.

In order to separate a quantity into two factors, one of which is a perfect power of any given degree, it is necessary to ascertain if the quantity contains a numerical factor that is a perfect power of that degree. To do this we must see if the quantity is divisible by any of the perfect powers of that degree. Thus, if the radical is of the third degree the perfect powers to be tried as divisors are 8, 27, 64, 125, 216, 343, 512, 729, &c. If the radical is of the fourth degree, the divisors are 16, 81, 256, 625, &c. If the radical is of the fifth degree, the divisors are $32=2^5$, $243=3^5$, $1024=4^5$, and so on.

(4) $\sqrt[3]{\tfrac{1}{2}}=\sqrt[3]{\tfrac{1}{2}\times\tfrac{4}{4}}=\sqrt[3]{\tfrac{1}{8}\times 4}=\tfrac{1}{2}\sqrt[3]{4}$;

$\sqrt[3]{\tfrac{3}{4}}=\sqrt[3]{\tfrac{1}{8}\times 6}=\tfrac{1}{2}\sqrt[3]{6}$; $\sqrt[3]{\tfrac{1}{6}}=\sqrt[3]{\tfrac{1}{216}\times 36}=\tfrac{1}{6}\sqrt[3]{36}$;

$\sqrt[3]{\tfrac{5}{9}}=\sqrt[3]{\tfrac{1}{27}\times 15}=\tfrac{1}{3}\sqrt[3]{15}$; $\sqrt[3]{\tfrac{7}{8}}=\sqrt[3]{\tfrac{1}{8}\times 7}=\tfrac{1}{2}\sqrt[3]{7}$;

$\sqrt[3]{\tfrac{9}{25}}=\sqrt[3]{\tfrac{9}{25}\times\tfrac{5}{5}}=\sqrt[3]{\tfrac{1}{125}\times 45}=\tfrac{1}{5}\sqrt[3]{45}.$

(6) $\sqrt[4]{\dfrac{2}{3}}=\sqrt[4]{\dfrac{2}{3}\times\dfrac{3^3}{3^3}}=\sqrt[4]{\dfrac{1}{3^4}\times 2\times 3^3}=\dfrac{1}{3}\sqrt[4]{2\times 3^3}=\dfrac{1}{3}\sqrt[4]{54}.$

(7) $\sqrt[5]{64}=\sqrt[5]{32\times 2}=2\sqrt[5]{2}$; $\sqrt[5]{729a^6}=\sqrt[5]{243a^5\times 3a}=3a\sqrt[5]{3a}$,

$\sqrt[6]{\dfrac{1}{2}}=\sqrt[6]{\dfrac{1}{2}\times\dfrac{2^5}{2^5}}=\sqrt[6]{\dfrac{1}{2^6}\times 2^5}=\dfrac{1}{2}\sqrt[6]{2^5}=\dfrac{1}{2}\sqrt[6]{32}$;

$\sqrt[6]{\dfrac{2}{3}}=\sqrt[6]{\dfrac{2}{3}\times\dfrac{3^5}{3^5}}=\sqrt[6]{\dfrac{1}{3^6}\times 2\times 3^5}=\dfrac{1}{3}\sqrt[6]{2\times 3^5}=\dfrac{1}{3}\sqrt[6]{486}$;

$\sqrt[5]{\dfrac{3}{4}}=\sqrt[5]{\dfrac{3}{2^2}}=\sqrt[5]{\dfrac{3}{2^2}\times\dfrac{2^3}{2^3}}=\sqrt[5]{\dfrac{1}{2^5}\times 3\times 2^3}=\dfrac{1}{2}\sqrt[5]{24}.$

or thus $\sqrt[5]{\frac{3}{4}}=\sqrt[5]{\frac{3}{4}\times\frac{4^4}{4^4}}=\sqrt[5]{\frac{1}{4^5}\times 3\times 4^4}=\frac{1}{4}\sqrt[5]{768}.$

The first method has the advantage of giving the result in the most simple form.

ADDITION AND SUBTRACTION OF RADICALS.

Article 204.

(14) $2\sqrt[3]{\frac{1}{4}}=2\sqrt[3]{\frac{1}{8}\times 2}=\sqrt[3]{2}$; $8\sqrt[3]{\frac{1}{32}}=8\sqrt{\frac{1}{64}\times 2}=2\sqrt[3]{2}.$

$\therefore$ Sum $=3\sqrt[3]{2}.$

(17) $3\sqrt{\frac{2}{3}}=\sqrt{9\times\frac{2}{3}}=\sqrt{6}$; $7\sqrt{\frac{27}{50}}=7\sqrt{\frac{9\times 3}{25\times 2}\times\frac{2}{2}}=7\sqrt{\frac{9}{100}\times 6}$

$=\frac{21}{10}\sqrt{6}$; $-\sqrt{54}=-\sqrt{9\times 6}=-3\sqrt{6}$;

$\therefore\ \sqrt{6}+\frac{21}{10}\sqrt{6}-3\sqrt{6}=(1+\frac{21}{10}-3)\sqrt{6}=\frac{1}{10}\sqrt{6}.$

(18) $-\frac{1}{2}\sqrt{12}=-\frac{1}{2}\sqrt{4\times 3}=-\sqrt{3}$; $4\sqrt{27}=4\sqrt{9\times 3}=12\sqrt{3}$;

$-2\sqrt{\frac{3}{16}}=-2\sqrt{\frac{1}{16}\times 3}=-\frac{1}{2}\sqrt{3}.$

$\therefore\ 2\sqrt{3}-\sqrt{3}+12\sqrt{3}-\frac{1}{2}\sqrt{3}=(2-1+12-\frac{1}{2})\sqrt{3}$

$=\frac{25}{2}\sqrt{3}.$

(20) $\sqrt[4]{16}=2$, $\sqrt[3]{81}=\sqrt[3]{27\times 3}=3\sqrt[3]{3}$, $-\sqrt[3]{-512}=-\sqrt{-8^3}$

$=8$, $\sqrt[3]{192}=\sqrt[3]{64\times 3}=4\sqrt[3]{3}$, $-7\sqrt[6]{9}=-7\sqrt[3]{3}$;

$\therefore\ 2+3\sqrt[3]{5}+8+4\sqrt[3]{3}-7\sqrt[3]{3}=10.$

(21) $8(\frac{3}{4})^{\frac{1}{2}}=8\sqrt{\frac{1}{4}\times 3}=4\sqrt{3}$, $\frac{1}{2}\times 12^{\frac{1}{2}}=\frac{1}{2}\sqrt{4\times 3}=\sqrt{3}$,

$-\frac{4}{3}\times 27^{\frac{1}{2}}=-\frac{4}{3}\sqrt{9\times 3}=-4\sqrt{3}$, $-2(\frac{3}{16})^{\frac{1}{2}}=-2\sqrt{\frac{1}{16}\times 3}$

$=-\frac{1}{2}\sqrt{3}$;

$\therefore\ 4\sqrt{3}+\sqrt{3}-4\sqrt{3}-\frac{1}{2}\sqrt{3}=(4+1-4-\frac{1}{2})\sqrt{3}=\frac{1}{2}\sqrt{3}$

23) $\sqrt{\frac{ab^3}{c^2}}=\sqrt{\frac{b^2}{c^2}\times ab}=\frac{b}{c}\sqrt{ab}$;

$$\frac{1}{2c}\sqrt{(a^3b-4a^2b^2+4ab^3)}=\frac{1}{2c}\sqrt{ab(a^2-4ab+4b^2)}=\frac{a-b}{2c}\sqrt{ab}.$$

$$\frac{b}{c}\sqrt{ab}+\frac{a-b}{2c}\sqrt{ab}=\frac{a}{2c}\sqrt{ab}.$$

MULTIPLICATION AND DIVISION OF RADICALS

Article 205.

(9) $\sqrt{3}=\sqrt[6]{3^3}$, $\sqrt[3]{2}=\sqrt[6]{2^2}$; $\sqrt[6]{3^3}\times\sqrt[6]{2^2}=\sqrt[6]{3^3\times2^2}=\sqrt[6]{108}$.

(10) $\sqrt[3]{b}=\sqrt[12]{b^4}$, $\sqrt[4]{a}=\sqrt[12]{a^3}$; $3\sqrt[12]{b^4}\times4\sqrt[12]{a^3}=12\sqrt[12]{a^3b^4}$.

(11) $\sqrt[2]{2}=\sqrt[12]{2^6}$, $\sqrt[3]{3}=\sqrt[12]{3^4}$, $\sqrt[4]{5}=\sqrt[12]{5^3}$;

$\sqrt[12]{2^6}\times\sqrt[12]{3^4}\times\sqrt[12]{5^3}=\sqrt[12]{2^6\times3^4\times5^3}=\sqrt[12]{64\times81\times125}$

$=\sqrt[12]{648000}$.

(12) $\sqrt{2}\times\sqrt[3]{3}=\sqrt[6]{2^3}\times\sqrt[6]{3^2}=\sqrt[6]{2^3\times3^2}=\sqrt[12]{2^6\times3^4}$;

$\sqrt[4]{\frac{1}{2}}\times\sqrt[3]{\frac{1}{3}}=\sqrt[12]{\frac{1}{2^3}}\times\sqrt[12]{\frac{1}{3^4}}=\sqrt[12]{\frac{1}{2^3\times3^4}}$;

$\sqrt[12]{2^6\times3^4}\times\sqrt[12]{\frac{1}{2^3\times3^4}}=\sqrt[12]{2^3}=\sqrt[4]{2}$.

Or thus, $\sqrt{2}\times\sqrt[3]{3}\times\sqrt[4]{\frac{1}{2}}\times\sqrt[3]{\frac{1}{3}}=\sqrt[4]{2^2}\times\sqrt[4]{\frac{1}{2}}\times\sqrt[3]{3}\times\sqrt[3]{\frac{1}{3}}$

$=\sqrt[4]{2}$, Since $2^2\times\frac{1}{2}=2$, and $3\times\frac{1}{3}=1$.

(13) $\sqrt[n]{x^2}=\sqrt[2n]{x^4}$, $\sqrt[3n]{x^3}=\sqrt[n]{x}=\sqrt[2n]{x^2}$

$\sqrt[2n]{x}\times\sqrt[2n]{x^4}\times\sqrt[2n]{x^2}=\sqrt[2n]{x\times x^4\times x^2}=\sqrt[2n]{x^7}$.

(18) $\sqrt[6]{72}\div\sqrt{2}=\sqrt[6]{72}\div\sqrt[6]{8}=\sqrt[6]{9}=\sqrt[3]{\sqrt{9}}=\sqrt[3]{3}$.

(19) $4\sqrt[3]{9}\div2\sqrt{3}=4\sqrt[6]{9^2}\div2\sqrt[6]{3^3}=2\sqrt[6]{3}$.

(20) $20\sqrt[6]{200}\div4\sqrt{2}=20\sqrt[6]{200}\div4\sqrt[6]{8}=5\sqrt[6]{25}=5\sqrt[3]{5}$.

(21) $\sqrt[6]{72}\div\sqrt[3]{3}=\sqrt[6]{72}\div\sqrt[6]{9}=\sqrt[6]{8}=\sqrt{\sqrt[3]{8}}=\sqrt{2}$.

(22) $\sqrt[3]{4}\div\frac{1}{2}\sqrt[3]{6}=2\sqrt[3]{\frac{2}{3}}=2\sqrt[3]{\frac{2}{3}\times\frac{9}{9}}=2\sqrt[3]{\frac{1}{27}\times18}=\frac{2}{3}\sqrt[3]{18}$.

Or thus, $\frac{1}{2}\sqrt[3]{6}=\sqrt[3]{\frac{1}{8}\times6}=\sqrt[3]{\frac{3}{4}}$; $\sqrt[3]{4}\div\sqrt[3]{\frac{3}{4}}=\sqrt[3]{\frac{16}{3}}$

$=\sqrt[3]{\frac{8\times2}{3}\times\frac{9}{9}}=\sqrt[3]{\frac{8}{27}\times18}=\frac{2}{3}\sqrt[3]{18}.$

(23) $\sqrt{\frac{b}{a}}=\sqrt[4]{\frac{b^2}{a^2}}$; $\sqrt[4]{\frac{b^2}{a^2}}\div\sqrt[4]{\frac{a}{b}}=\sqrt[4]{\frac{b^2}{a^2}\times\frac{a}{b}}=\sqrt[4]{\frac{b}{a}}.$

(24) $\frac{1}{2}\sqrt{\frac{1}{2}}=\frac{1}{2}\sqrt{\frac{1}{4}\times2}=\frac{1}{4}\sqrt{2}.$

$\sqrt{2}+3\sqrt{\frac{1}{2}}=\sqrt{2}+3\sqrt{\frac{1}{4}\times2}=\sqrt{2}+\frac{3}{2}\sqrt{2}=\frac{5}{2}\sqrt{2}$

$\frac{1}{4}\sqrt{2}\div\frac{5}{2}\sqrt{2}=\frac{1}{4}\times\frac{2}{5}=\frac{1}{10}.$

(25)
$$\begin{array}{l} 3+\sqrt{5} \\ 2-\sqrt{5} \\ \hline 6+2\sqrt{5} \\ \quad -3\sqrt{5}-5 \\ \hline 1-\sqrt{5}. \end{array}$$

(26)
$$\begin{array}{l} \sqrt{2}+1 \\ \sqrt{2}-1 \\ \hline 2+\sqrt{2} \\ \quad -\sqrt{2}-1 \\ \hline 2-1=1. \end{array}$$

(27)
$$\begin{array}{l} 11\sqrt{2}-4\sqrt{15} \\ \sqrt{6}+\sqrt{5} \\ \hline 11\sqrt{12}-4\sqrt{90} \\ \qquad +11\sqrt{10}-4\sqrt{75} \\ \hline 11\sqrt{12}-\sqrt{10}-4\sqrt{75} \\ =22\sqrt{3}-\sqrt{10}-20\sqrt{3} \\ =2\sqrt{3}-\sqrt{10} \end{array}$$

Observe that $-4\sqrt{90}=-12\sqrt{10}$

(28)
$$\begin{array}{l} \sqrt{2}+\sqrt{3}. \\ \sqrt{2}+\sqrt{3}. \\ \hline 2+\sqrt{6}. \\ \quad +\sqrt{6}+3 \\ \hline 5+2\sqrt{6} \\ 5+2\sqrt{6} \\ \hline 25+10\sqrt{6} \\ \quad +10\sqrt{6}+4\times6 \\ \hline 49+20\sqrt{6}. \end{array}$$

(29) $3\sqrt{4+6\sqrt{2}}\times5\sqrt{2}=15\sqrt{4\times2+12\sqrt{2}}$

$=15\sqrt{4\times2+4\times3\sqrt{2}}=15\times2\sqrt{2+3\sqrt{2}}=30\sqrt{2+3\sqrt{2}}$

(30) $\sqrt[3]{12+\sqrt{19}}\times\sqrt[3]{12-\sqrt{19}}=\sqrt[3]{\{(12+\sqrt{19})(12-\sqrt{19})\}}$

$=\sqrt[3]{\{144-19\}}=\sqrt[3]{144-19}=\sqrt[3]{125}=5.$

(33) $2\sqrt{8}=4\sqrt{2}$, $\sqrt{72}=6\sqrt{2}$, $5\sqrt{20}=10\sqrt{2}$;

$(2\sqrt{8}+3\sqrt{5}-7\sqrt{2})=3\sqrt{5}-3\sqrt{2}$;

$(\sqrt{72}-5\sqrt{20}-2\sqrt{2})=4\sqrt{2}-10\sqrt{5}$

$$12\sqrt{10}-24$$

$$-150+30\sqrt{10}$$

$$42\sqrt{10}-174. \quad \textit{Ans.}$$

Article 206.

(4) Multiply both terms by 4^2 or 16.

(5) $\frac{8-5\sqrt{2}}{3-2\sqrt{2}}=\frac{8-5\sqrt{2}}{3-2\sqrt{2}}\times\frac{3+2\sqrt{2}}{3+2\sqrt{2}}=\frac{4+\sqrt{2}}{9-8}=4+\sqrt{2}.$

(6) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}=\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\times\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\frac{5+2\sqrt{6}}{3-2}=5+2\sqrt{6}.$

(7) $\frac{\sqrt{3}+1}{2-\sqrt{3}}=\frac{\sqrt{3}+1}{2-\sqrt{3}}\times\frac{2+\sqrt{3}}{2+\sqrt{3}}=\frac{5+3\sqrt{3}}{4-3}=5+3\sqrt{3}.$

(8) $\frac{1-\sqrt{5}}{3+\sqrt{5}}=\frac{1-\sqrt{5}}{3+\sqrt{5}}\times\frac{3-\sqrt{5}}{3-\sqrt{5}}=\frac{8-4\sqrt{5}}{9-5}=2-\sqrt{5}.$

(9) $\frac{3\sqrt{5}-2\sqrt{2}}{2\sqrt{5}-\sqrt{18}}\times\frac{2\sqrt{5}+\sqrt{18}}{2\sqrt{5}+\sqrt{18}}=\frac{18+5\sqrt{10}}{20-18}=9+\frac{5}{2}\sqrt{10}$

(10) Multiply both terms by $\sqrt{2}+\sqrt{3}+\sqrt{5}$, and the fraction becomes $\frac{\sqrt{2}+\sqrt{3}+\sqrt{5}}{2\sqrt{6}}$; then multiply both terms by $\sqrt{6}$ and it becomes $\frac{2\sqrt{3}+3\sqrt{2}+\sqrt{30}}{12}$.

(11) $\frac{3+4\sqrt{3}}{\sqrt{6}+\sqrt{2}-\sqrt{5}}\times\frac{\sqrt{6}+\sqrt{2}+\sqrt{5}}{\sqrt{6}+\sqrt{2}+\sqrt{5}}$

$=\frac{(3+4\sqrt{3})(\sqrt{6}+\sqrt{2}+\sqrt{5})}{3+4\sqrt{3}}=\sqrt{6}+\sqrt{2}+\sqrt{5}.$ *Ans.*

10

(12) $\frac{1}{x+\sqrt{x^2-1}}\times\frac{x-\sqrt{x^2-1}}{x-\sqrt{x^2-1}}=\frac{x-\sqrt{x^2-1}}{x^2-(x^2-1)}=x-\sqrt{x^2-1}$;

$\frac{1}{x-\sqrt{x^2-1}}\times\frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}=\frac{x+\sqrt{x^2-1}}{x^2-(x^2-1)}=x+\sqrt{x^2-1}$.

Sum $=2x$. *Ans.*

(13) Multiply both terms by $\sqrt{(x+a)}+\sqrt{(x-a)}$ and the fraction becomes $\frac{(x+a)+2\sqrt{(x^2-a^2)}+(x-a)}{(x+a)-(x-a)}$

$=\frac{x+\sqrt{(x^2-a^2)}}{a}$.

(14) $\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}\times\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}=x^2+\sqrt{x^4-1}$

$\frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}\times\frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}=x^2-\sqrt{x^4-1}$

Sum $=2x^2$.

(16) $\frac{1}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}=\frac{2-\sqrt{3}}{4-3}=2-\sqrt{3}=.267949+$.

(17) $\frac{1+\sqrt{2}}{2-\sqrt{2}}\times\frac{2+\sqrt{2}}{2+\sqrt{2}}=\frac{4+3\sqrt{2}}{4-2}=2+\frac{3}{2}\sqrt{2}=4.12132+$.

(18) $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}\times\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}=\frac{16+2\sqrt{60}}{2}$

$=8+2\sqrt{15}=15.745966+$.

IMAGINARY, OR IMPOSSIBLE QUANTITIES.

Article 210.

(3) $(a\sqrt{-1})^3=(a\sqrt{-1})^2\times(a\sqrt{-1})=-a^2\times a\sqrt{-1}$

$=-a^3\sqrt{-1}$.

$(a\sqrt{-1})^4=(a\sqrt{-1})^2\times(a\sqrt{-1})^2=(-a^2)\times(-a^2)=a^4$

(4) $(2\sqrt{-3})\times(3\sqrt{-2})=(2\sqrt{3}\sqrt{-1})\times(3\sqrt{2}\sqrt{-1})$
$=-6\sqrt{6}.$

(5) $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}=\frac{-1+\sqrt{-3}}{2}$

$$\begin{array}{l} -1+\sqrt{-3} \\ -1+\sqrt{-3} \\ \hline 1-\sqrt{-3} \\ \quad -\sqrt{-3}-3 \\ \hline -2-2\sqrt{-3} \\ -1+\ \sqrt{-3} \\ \hline 2+2\sqrt{-3} \\ \quad -2\sqrt{-3}-2(-3) \\ \hline 2 \qquad\quad +6=8\ ; \end{array} \qquad \begin{array}{l} -1-\sqrt{-3} \\ -1-\sqrt{-3} \\ \hline 1+\sqrt{-3} \\ \quad +\sqrt{-3}+(-3) \\ \hline -2+2\sqrt{-3} \\ -1-\ \sqrt{-3} \\ \hline 2-2\sqrt{-3} \\ \quad +2\sqrt{-3}-2(-3) \\ \hline 2 \qquad\quad +6=8\ ; \end{array}$$

$2^3=8$, and $\frac{8}{8}=1$. *Ans.* $\qquad \frac{8}{8}=1$. *Ans.*

(6) $6\sqrt{-3}=6\sqrt{3}\sqrt{-1}$; $2\sqrt{-4}=2\sqrt{4}\sqrt{-1}=4\sqrt{-1}$;
$6\sqrt{3}\sqrt{-1}\div4\sqrt{-1}=\frac{6\sqrt{3}\sqrt{-1}}{4\sqrt{-1}}=\frac{3}{2}\sqrt{3}.$

(7) $\frac{1+\sqrt{-1}}{1-\sqrt{-1}}\times\frac{1+\sqrt{-1}}{1+\sqrt{-1}}=\frac{1+2\sqrt{-1}-1}{1-(-1)}=\frac{2\sqrt{-1}}{2}=\sqrt{-1}.$

(8) $(x+a\sqrt{-1})\times(x-a\sqrt{-1})=x^2-a^2(-1)=x^2+a^2$;
$(x+a)\times(x-a)=x^2-a^2$; $(x^2+a^2)(x^2-a^2)=x^4-a^4$.

(9) Multiply the quantities together.

MULTIPLICATION AND DIVISION OF QUANTITIES WITH FRACTIONAL EXPONENTS.

Article 213.

(4)
$$\begin{array}{l} a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}} \\ a^{\frac{1}{3}}-b^{\frac{1}{3}} \\ \hline a+a^{\frac{2}{3}}b^{\frac{1}{3}}+a^{\frac{1}{3}}b^{\frac{2}{3}} \\ \quad -a^{\frac{2}{3}}b^{\frac{1}{3}}-a^{\frac{1}{3}}b^{\frac{2}{3}}-b \\ \hline \quad a-b \quad \textit{Ans.} \end{array}$$

(5)
$$\begin{array}{l} x^{\frac{1}{4}}y+y^{\frac{2}{3}} \\ \quad x^{\frac{1}{4}}-y^{-\frac{1}{3}} \\ \hline x^{\frac{1}{2}}y+x^{\frac{1}{4}}y^{\frac{2}{3}} \\ \qquad -x^{\frac{1}{4}}y^{\frac{2}{3}}-y^{\frac{1}{3}} \\ \hline x^{\frac{1}{2}}y-y^{\frac{1}{3}}. \quad \textit{Ans.} \end{array}$$

(6) $(a+b)^{\frac{1}{m}}\times(a-b)^{\frac{1}{m}}=[(a+b)(a-b)]^{\frac{1}{m}}=(a^2-b^2)^{\frac{1}{m}}$;
$(a+b)^{\frac{1}{n}}\times(a-b)^{\frac{1}{n}}=[(a+b)(a-b)]^{\frac{1}{n}}=(a^2-b^2)^{\frac{1}{n}}$;
$(a^2-b^2)^{\frac{1}{m}}\times(a^2-b^2)^{\frac{1}{n}}=(a^2-b^2)^{\frac{1}{m}+\frac{1}{n}}=(a^2-b^2)^{\frac{m+n}{mn}}$.

(7) Observe that $\frac{2}{3}-\frac{1}{4}=\frac{5}{12}$; $\frac{3}{m}-\frac{2}{n}=\frac{3n-2m}{mn}$.

(8)
$$\begin{array}{l|l}
a^{\frac{3}{4}}-b^{\frac{3}{4}} & a^{\frac{1}{4}}-b^{\frac{1}{4}} \\
\underline{a^{\frac{3}{4}}-a^{\frac{1}{2}}b^{\frac{1}{4}}} & a^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}. \quad \textit{Ans.} \\
+a^{\frac{1}{2}}b^{\frac{1}{4}} & \\
\underline{a^{\frac{1}{2}}b^{\frac{1}{4}}-a^{\frac{1}{4}}b^{\frac{1}{2}}.} & \\
a^{\frac{1}{4}}b^{\frac{1}{2}}-b^{\frac{3}{4}} & \\
\underline{a^{\frac{1}{4}}b^{\frac{1}{2}}-b^{\frac{3}{4}}.} &
\end{array}$$

(9)
$$\begin{array}{l|l}
a-b^2 & a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{4}}b+b^{\frac{3}{2}} \\
\underline{a+a^{\frac{3}{4}}b^{\frac{1}{2}}+a^{\frac{1}{2}}b+a^{\frac{1}{4}}b^{\frac{3}{2}}} & a^{\frac{1}{4}}-b^{\frac{1}{2}} \quad \textit{Ans.} \\
-a^{\frac{3}{4}}b^{\frac{1}{2}}-a^{\frac{1}{2}}b-a^{\frac{1}{4}}b^{\frac{3}{2}}-b^2 & \\
\underline{-a^{\frac{3}{4}}b^{\frac{1}{2}}-a^{\frac{1}{2}}b-a^{\frac{1}{4}}b^{\frac{3}{2}}-b^2.} &
\end{array}$$

POWERS AND ROOTS OF QUANTITIES WITH FRACTIONAL EXPONENTS.

Article 215.

(5) $a^0=1$, $x^0=1$, (Art. 82).

$$\begin{array}{l}
a^{\frac{1}{3}}x^{-1}+a^{-\frac{1}{3}}x \\
\underline{a^{\frac{1}{3}}x^{-1}+a^{-\frac{1}{3}}x} \\
a^{\frac{2}{3}}x^{-2}+a^0x^0 \\
\underline{\qquad\qquad +a^0x^0+a^{-\frac{2}{3}}x^2} \\
a^{\frac{2}{3}}x^{-2}+2 \quad +a^{-\frac{2}{3}}x^2 \\
\underline{a^{\frac{1}{3}}x^{-1}+a^{-\frac{1}{3}}x} \\
ax^{-3}+2a^{\frac{1}{3}}x^{-1}+a^{-\frac{1}{3}}x \\
\underline{\qquad +a^{\frac{1}{3}}x^{-1}+2a^{-\frac{1}{3}}x+a^{-1}x^3} \\
ax^{-3}+3a^{\frac{1}{3}}x^{-1}+3a^{-\frac{1}{3}}x+a^{-1}x^3.
\end{array}$$

(6) $[3(5)^{\frac{1}{3}}]^{\frac{1}{2}}=3^{\frac{1}{2}}\times5^{\frac{1}{6}}=3^{\frac{3}{6}}\times5^{\frac{1}{6}}=(3^3\times5)^{\frac{1}{6}}=(135)^{\frac{1}{6}}$.

$$\left\{\frac{7a^2(a)^{\frac{1}{3}}}{9(343b^2)^{\frac{1}{6}}}\right\}^{\frac{1}{2}}=\left\{\frac{7a^2(a)^{\frac{1}{3}}}{3^2(7^3b^2)^{\frac{1}{6}}}\right\}^{\frac{1}{2}}=\frac{7^{\frac{1}{2}}a(a)^{\frac{1}{6}}}{3(7^{\frac{1}{4}}b^{\frac{1}{6}})}=\frac{7^{\frac{1}{4}}a^{\frac{7}{6}}}{3b^{\frac{1}{6}}}$$

(8)

$$5x^3-4x(5cx)^{\frac{1}{2}}+4c \;\big|\; 5^{\frac{1}{2}}x^{\frac{3}{2}}-2c^{\frac{1}{2}}. \quad \textit{Ans.}$$

$$5x^3$$

$$2(5^{\frac{1}{2}}x^{\frac{3}{2}})-2c^{\frac{1}{2}} \;\big|\; -4x(5cx)^{\frac{1}{2}}+4c$$

$$-4x(5cx)^{\frac{1}{2}}+4c.$$

(9) Separating the third term into its parts, and arranging the terms according to the ascending powers of a, we have

$$1-\frac{3}{2}a^{\frac{1}{2}}+\frac{41}{16}a-\frac{3a^{\frac{3}{2}}}{2}+a^2 \;\Big|\; 1-\frac{3a^{\frac{1}{2}}}{4}+a.=\text{sq. root}$$

$$1$$

$$2-\frac{3a^{\frac{1}{2}}}{4} \;\Big|\; -\frac{3}{2}a^{\frac{1}{2}}+\frac{41}{16}a$$

$$-\frac{3}{2}a^{\frac{1}{2}}+\frac{9a}{16}$$

$$2-\frac{3a^{\frac{1}{2}}}{2}+a \;\Big|\; 2a-\frac{3a^{\frac{3}{2}}}{2}+a^2$$

$$2a-\frac{3a^{\frac{3}{2}}}{2}+a^2$$

(10)

$$\tfrac{1}{8}a^3-\tfrac{3}{2}a^2b^{\frac{1}{2}}+6ab-8b^{\frac{3}{2}} \;\big|\; \tfrac{1}{2}a-2b^{\frac{1}{2}}=\text{cube root.}$$

$$\tfrac{1}{8}a^3$$

$$\tfrac{3}{4}a^2-3ab^{\frac{1}{2}}+4b \;\big|\; -\tfrac{3}{2}a^2b^{\frac{1}{2}}+6ab-8b^{\frac{3}{2}}$$

$$-\tfrac{3}{2}a^2b^{\frac{1}{2}}+6ab-8b^{\frac{3}{2}}$$

EQUATIONS CONTAINING RADICALS.

Article 216.

(4) Transpose 3 and then square both sides.

(5) Square both sides, transpose 1, and square again.

(6) Square both sides, omit x on each side, divide both sides by $2a$, transpose $\sqrt{x}$, or $\frac{a}{2}$ and square. The answer is either $\frac{(a-1)^2}{4}$ or $\frac{(1-a)^2}{4}$, the two being equal to each other.

(7) Square both sides, transpose $2x-3a+2x$, divide by 2 and square again.

(8) Square both sides and transpose 13; square again and transpose 7; square again and transpose 3; whence $\sqrt{x}=1$ and $x=1$.

(9) Multiply both sides by the first term, transpose $2+x$ and square both sides.

(10) Multiply both sides by $\sqrt{x}$, transpose $\sqrt{a}$, then square both sides, and omit x^2 on each side.

(11) Transpose the second term to the left member and square both sides, omit x on each side, transpose the known quantities to the left side and square again.

(12) $a\sqrt{x}+b\sqrt{x}-c\sqrt{x}=d$, or $(a+b-c)\sqrt{x}=d$; whence $\sqrt{x}=\frac{d}{(a+b-c)}$, and $x=\frac{d^2}{(a+b-c)^2}$.

(13) Multiply both sides by $x\sqrt{x}$ to clear the equation of fractions, then divide by x and we have $(1-a)x=1$. The equation may also be cleared of fractions by multiplying both sides by x.

(14) Square both sides, omit a^2 on each side, then divide by x and square again.

(15) Since $x-4=(\sqrt{x}+2)(\sqrt{x}-2)$ the first member becomes $\sqrt{x}-2$; then by transposing we have $6=5\frac{1}{2}\sqrt{x}$, or $11\sqrt{x}=12$, whence $x=\frac{144}{121}$.

(16) Since $x-a=(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a})$ the first member becomes $\sqrt{x}-\sqrt{a}$; then by clearing of fractions and reducing we find $\sqrt{x}=4\sqrt{a}$, whence $x=16a$.

(17) Since $3x-1=(\sqrt{3x}+1)(\sqrt{3x}-1)$, the first member becomes $\sqrt{3x}-1$; then by clearing of fractions, reducing and squaring, x is found $=3$.

(18) $\sqrt{4a+x}=2\sqrt{b+x}-\sqrt{x}$,

$4a+x=4b+4x-4\sqrt{bx+x^2}+x$, by squaring,

$\sqrt{bx+x^2}=(b-a)+x$, by transposing and reducing,

$bx+x^2=(b-a)^2+2(b-a)x+x^2$, by squaring,

$(2a-b)x=(b-a)^2$, by transposing,

$$x=\frac{(b-a)^2}{2a-b}.$$

(19) $\sqrt{\frac{b}{a+x}}+\sqrt{\frac{c}{a-x}}=\sqrt[4]{\frac{4bc}{a^2-x^2}}.$

$\frac{b}{a+x}+2\sqrt{\frac{bc}{a^2-x^2}}+\frac{c}{a-x}=\sqrt{\frac{4bc}{a^2-x^2}}$ by squaring;

But $2\sqrt{\frac{bc}{a^2-x^2}}=\sqrt{\frac{4bc}{a^2-x^2}}$;

$\therefore\ \frac{b}{a+x}+\frac{c}{a-x}=0$, whence $x=\frac{a(b+c)}{b-c}$.

(20) Multiplying both terms of the first member by the numerator and then clearing of fractions and transposing, we have

$2\sqrt{x^2+ax}=a\,(c-1)-2x$,

$4x^2+4ax=a^2(c-1)^2-4ax(c-1)+4x^2$ by squaring,

$4acx=a^2(c-1)^2$ by reducing and transposing;

$$\therefore\ x=\frac{a(c-1)^2}{4c}.$$

(21) $\sqrt{\sqrt{x}+3}-\sqrt{\sqrt{x}-3}=\sqrt{2\sqrt{x}}.$

Squaring both sides, and observing that $\sqrt{\sqrt{x}+3}$ multiplied by $\sqrt{\sqrt{x}-3}$ produces $\sqrt{x-9}$, we have

$\sqrt{x}+3-2\sqrt{x-9}+\sqrt{x}-3=2\sqrt{x}$.

Reducing, and omitting $2\sqrt{x}$ on each side, we have

$-2\sqrt{x-9}=0$, and $4(x-9)=0$ by squaring,

whence $x=9$.

(22) Square both members, omit $\frac{1}{a^2}$ on each side, square again and omit $\frac{1}{x^4}$ on both sides; then multiply both members by x^2, clear the equation of fractions and the value of x is readily found.

23) Square both members, omit equal quantities on each side, place all the terms not under the radical on the right side and divide by 2, and we have

$\sqrt{\{(1-a^2)^2+2x(1+3a^2)+(1-a^2)x^2\}}=(a^2-1)-x$;

square both sides, and we have

$$(1-a^2)^2+2x(1+3a^2)+(1-a^2)x^2=(a^2-1)^2-2x(a^2-1)+x^2.$$

The square of $1-a^2$ is the same as the square of a^2-1, omitting these and x^2 on each side, and dividing by x and transposing

we have $a^2x=8a^2$;

whence $x=8$.

EXAMPLES IN INEQUALITIES.

NOTE. The subject of inequalities, though interesting and highly important in itself, is not much used in the subsequent parts of Algebra The last eight examples, that is, from the 10th to the close, may be regarded as so many independent algebraic theorems, the study of w may be omitted by all except the higher class of students.

Article 223.

) Squaring both quantities, subtracting 19 from each, an dividing by 2, we have

$\sqrt{70}>$, or $<1+3\sqrt{6}$;

$70>$, or $<1+6\sqrt{6}+54$, by squaring;

$15>$, or $<6\sqrt{6}$, by subtracting 55 from each member

$5>$, or $<2\sqrt{6}$, by dividing by 3;

$25>24$, by squaring;

hence $\sqrt{5}+\sqrt{14}$ is greater than $\sqrt{3}+3\sqrt{2}$.

(8) Multiplying both members of the first comparison by 12, to clear it of fractions, and reducing, we get $x<6$

Treating the second comparison in the same manner, we find $x>4$; hence if x is a whole number and is greater than 4 and less than 6, it must be 5.

(9) $2x+7$ not >19, $3x-5$ not <13,
or $2x$ not >12, $3x$ not <18,
x not >6. x not <6.

Hence if x is neither less nor greater than 6, it must be 6.

(10) In example 6, page 175 of the Algebra, it is shown that $a^2+b^2>2ab$.
Let $a=n$, and $b=1$, then by substitution
$n^2+1>2n$,
$n^2-n+1>n$, by subtracting n from each side
$n^3+1>n(n+1)$ by multiplying both sides by $n+1$, or
$n^3+1>n^2+n$.

(11) Referring again to example 6 we have
$a^2+b^2>2ab$,
$\frac{a^2}{ab}+\frac{b^2}{ab}>2$, by dividing by ab.
$\frac{a}{b}+\frac{b}{a}>2$, by reducing.

(12) If $x>y$ then $\sqrt{x}>\sqrt{y}$, and $\sqrt{xy}>\sqrt{y}\times\sqrt{y}$, or y,

$$\sqrt{xy}>y,$$
$$2\sqrt{xy}>2y;$$

$\therefore$ (Art. 222) $-2y>-2\sqrt{xy}$, add $x+y$ to each member
then $x-y>x-2\sqrt{xy}+y$,
or $x-y>(\sqrt{x}-\sqrt{y})^2$.

(13) Referring again to example 6, we have
$a^2+b^2>2ab$, subtract ab from each member;
$a^2-ab+b^2>ab$, multiply each side by $a+b$, we have
$a^3+b^3>a^2b+ab^2$, divide each side by a^2b^2,
$\frac{a^3}{a^2b^2}+\frac{b^3}{a^2b^2}>\frac{a^2b}{a^2b^2}+\frac{ab^2}{a^2b^2}$, or reducing,
$\frac{a}{b^2}+\frac{b}{a^2}>\frac{1}{b}+\frac{1}{a}$.

(14) From question 6, we have

$a^2+b^2>2ab$,

also $a^2+c^2>2ac$,

and $b^2+c^2>2bc$,

from which, by adding together the corresponding members, and dividing each member by 2, we have

$a^2+b^2+c^2>ab+ac+bc$.

(15) $\frac{a^3+b^3}{a^2+b^2}>$, or $<\frac{a^2+b^2}{a+b}$; multiply both members by $a+b$,

$\frac{a^4+ab^3+a^3b+b^4}{a^2+b^2}>$, or $<a^2+b^2$; multiply both members by a^2+b^2,

$a^4+ab^3+a^3b+b^4>$, or $<a^4+2a^2b^2+b^4$;

Subtracting a^4+b^4 from each member we have

$ab^3+a^3b>$, or $<2a^2b^2$; divide by ab,

$b^2+a^2>$, or $<2ab$.

But it has already been proved in example 6 that a^2+b^2 is $>2ab$, when a is not equal to b; therefore,

$\frac{a^3+b^3}{a^2+b^2}>\frac{a^2+b^2}{a+b}$.

Or thus, $a^2+b^2>2ab$, multiply both sides by ab,

$a^3b+ab^3>2a^2b^2$, add a^4+b^4 to each member,

$a^4+a^3b+ab^3+b^4>a^4+2a^2b^2+b^4$;

dividing each member by $a+b$, and then by a^2+b^2, we have

$\frac{a^3+b^3}{a^2+b^2}>\frac{a^2+b^2}{a+b}$, which establishes the proposition.

(16) $x^2=a^2+b^2$, and $y^2=c^2+d^2$;

$x^2y^2=(a^2+b^2)(c^2+d^2)=a^2c^2+a^2d^2+b^2c^2+b^2d^2$,

$(ac+bd)^2= \quad . \quad . \quad . \quad . \quad a^2c^2+2abcd+b^2d^2$,

$x^2y^2-(ac+bd)^2=a^2d^2-2abcd+b^2c^2=(ad-bc)^2$,

but $x^2y^2-(ac+bd)^2=\{xy+(ac+bd)\}\{xy-(ac+bd)\}$;

divide each member by $xy+(ac+bd)$, and we have

$xy-(ac+bd)=\frac{(ad-bc)^2}{xy+ac+bd}$.

But the second member of this equation is necessarily positive since the numerator is a square and the denominator positive hence the first member is positive; that is, $xy>ac+bd$.

(17) $a^2 > a^2 - (b-c)^2$, since $(b-c)^2$ is necessarily positive,
$> (a+b-c)(a+c-b)$ by factoring ;
$b^2 > b^2 - (a-c)^2$,
$> (a+b-c)(b+c-a)$;
$c^2 > c^2 - (a-b)^2$,
$> (a+c-b)(b+c-a)$.
Multiplying together the corresponding members of these inequalities, $a^2b^2c^2 > (a+b-c)^2(a+c-b)^2(b+c-a)^2$; extracting the square root of both members we have $abc > (a+b-c)(a+c-b)(b+c-a)$.

EQUATIONS OF THE SECOND DEGREE.

INCOMPLETE EQUATIONS.

Article 228.

(12) Multiply both members by $\sqrt{a^2+x^2}$, transpose a^2+x^2 and square again.

(13) Multiply both members by bx, transpose ab and then square each member.

(14) Multiply both members by the product of the denominators and reduce.

(15) Multiplying both members by the denominator of the first transposing and factoring, we have

$a(1-b) = (b+1)\sqrt{a^2-x^2}$;

$a^2(1-b)^2 = (b+1)^2(a^2-x^2)$, by squaring,

$(b+1)^2x^2 = 4a^2b$, by transposing and reducing ;

$$\therefore\ x^2 = \frac{4a^2b}{(b+1)^2}, \text{ and } x = \pm\frac{2a\sqrt{b}}{b+1}.$$

QUESTIONS PRODUCING INCOMPLETE EQUATIONS OF THE SECOND DEGREE

Article 229.

(2) Let $x=$ the number, then
$x^2 - 17 = 130 - 2x^2$.
Whence $x=7$.

(3) Let $x=$ the number, then

$(10-x)x=10(x-6\frac{2}{5})$.

Whence $x=8$.

(4) Let $x=$ the number, then

$30-\frac{1}{3}x^2=\frac{1}{4}x^2+9$.

Whence $x=6$.

(5) To avoid fractions let $9x=$ the greater, then $\frac{2}{9}$ of $9x=2x$ and $9x-2x=7x$, will represent the less;

$\therefore$ $(9x)^2-(7x)^2$, or $81x^2-49x^2=128$.

Whence $x=2$, $\therefore$ $9x=18$, and $7x=14$.

(7) Let $x=$ the greater number, then $14-x=$ the less;

then $\frac{x}{14-x} : \frac{14-x}{x} :: 16 : 9$;

whence $\frac{9x}{14-x}=\frac{16(14-x)}{x}$; clearing of fractions

$9x^2=16(14-x)^2$; extracting the square root

$3x=4(14-x)$;

Whence $x=8$, and $14-x=6$.

(8) Let $x=$ the number, then

$(20+x)(20-x)=319$.

Whence $x=9$.

(9) Let $x=$ the greater, then $\frac{126}{x}=$ the less, and

$x\div\frac{126}{x}=x\times\frac{x}{126}=\frac{x^2}{126}=3\frac{1}{2}$.

Whence $x^2=441$; $\therefore$ $x=21$ and $\frac{126}{x}=6$.

(10) Let $x=$ one of the numbers, then $\frac{p}{x}=$ the other, and

$x\div\frac{p}{x}=x\times\frac{x}{p}=\frac{x^2}{p}=q$.

Whence $x=\sqrt{pq}$; $\frac{p}{x}=\frac{p}{\sqrt{pq}}=\sqrt{\frac{p}{q}}$.

(11) Let $x=$ one of the numbers, then its square is x^2, and the square of the other is $370-x^2$.

$\therefore\ x^2-(370-x^2)=208.$

Whence $x=17$, and $370-x^2=370-289=81$.

$\therefore$ the other $=\sqrt{81}=9$.

(12) Let $x=$ one of the numbers, then its square $=x^2$, and the square of the other is $c-x^2$;

$\therefore\ x^2-(c-x^2)=d.$

$2x^2=c+d,$

$4x^2=2(c+d),$

$2x=\sqrt{2(c+d)}$

$x=\frac{1}{2}\sqrt{2(c+d)}$

$\sqrt{c-x^2}=\sqrt{c-\frac{1}{2}(c+d)}=\sqrt{\frac{1}{2}(c-d)}=\frac{1}{2}\sqrt{2(c-d)}.$

(13) Let $x=$ the sum, then $\frac{5x}{100}=$ interest for 1 year, and

$\frac{1}{4}$ of $\frac{5x}{100}=\frac{5x}{400}=$ interest for 3 months, or $\frac{1}{4}$ of a year

$\therefore\ x\times\frac{5x}{400}=720$, or $\frac{5x^2}{400}=720$;

$5x^2=720\times400,$

$x^2=144\times400,$

$x=12\times20=240.$

(14) Let $x=$ the first, then $\frac{a}{x}=$ the second, and $\frac{b}{x}=$ the third

$\therefore\ \frac{a^2}{x^2}+\frac{b^2}{x^2}=c$; whence $x=\sqrt{\left(\frac{a^2+b^2}{c}\right)}$

$a\div\sqrt{\left(\frac{a^2+b^2}{c}\right)}=a\sqrt{\left(\frac{c}{a^2+b^2}\right)}$; and $b\div\sqrt{\left(\frac{a^2+b^2}{c}\right)}$

$=b\sqrt{\left(\frac{c}{a^2+b^2}\right)}.$

(17) Let $x=$ number of drawers, then $x\times x=x^2$ the number of divisions, and $x^2\times4x=4x^3=5324$.

Whence $x^3=1331$, and $x=11$.

(18) The solution of this question involves a knowledge of two elementary principles of Natural Philosophy, with which the student should be rendered familiar by simple illustrations.

1st. *In uniform motion, the space divided by the time is equal to the velocity or rate of moving.*

2nd. *In uniform motion, the space divided by the velocity is equal to the time.*

Thus, if a man travels 80 miles in 4 days, his rate of traveling (velocity) is 20 miles per day. Or, if a man travels 100 miles at the rate of 20 miles a day, the time of traveling is 5 days.

Let $x=$ the distance B traveled, then

$x+18=$ " A " .

Then since the distance traveled, divided by the number of days, gives the number of miles traveled in one day, or the rate of traveling, we have

$\frac{x}{15\frac{3}{4}}$, or $\frac{4x}{63}=$ A's rate of traveling, and

$\frac{x+18}{28}=$ B's rate " .

But the distance traveled, divided by the rate of traveling, gives the time, therefore

$(x+18)\div\frac{4x}{63}=\frac{63(x+18)}{4x}=$ time A traveled, and

$x\div\frac{x+18}{28}=\frac{28x}{x+18}=$ " B " .

But since they both traveled the same time we have

$$\frac{63(x+18)}{4x}=\frac{28x}{x+18}.$$

Divide each side by 7 to reduce to lower terms,

$$\frac{9(x+18)}{4x}=\frac{4x}{x+18}.$$

Multiplying by $4x$ and $x+18$, and indicating the operations, we have

$9(x+18)^2=16x^2$;

Extracting the square root of both members,

$3(x+18)=4x$.

Whence $x=54$, and $x+18=72$;

and $54+72=126$, the required distance.

(19) The solution of this question involves principles analogous to the preceding.

Let $x=$ the number of days, then $x-4=$ days A worked, and $x-7=$ days B worked.

Also $\frac{75}{x-4}=$ A's daily wages, and $\frac{48}{x-7}=$ B's daily wages.

If B had played only 4 days he would have worked $x-4$ days, and would have received

$\left(\frac{48}{x-7}\right)(x-4)$ shillings.

If A had played 7 days he would have worked $x-7$ days, and would have received

$\left(\frac{75}{x-4}\right)(x-7)$ shillings. But by the question each would have received the same sum, therefore,

$$\left(\frac{75}{x-4}\right)(x-7)=\left(\frac{48}{x-7}\right)(x-4).$$

Multiplying each side by $x-4$ and $x-7$ to clear the equation of fractions, and indicating the multiplication, we have

$75(x-7)^2=48(x-4)^2$;

dividing by 3 to reduce it to lower terms

$25(x-7)^2=16(x-4)^2$;

extracting the square root of both members,

$5(x-7)=4(x-4)$.

Whence $x=19$.

(20) Let $\frac{1}{x}=$ the part of the wine drawn each time, then $\frac{1}{x}$ of 1 (the whole) is $\frac{1}{x}=$ the part drawn at the 1st draught, and

$1-\frac{1}{x}=\frac{x-1}{x}=$ the part remaining after the 1st draught.

$$\frac{x-1}{x}-\frac{1}{x}\text{ of }\frac{x-1}{x}=\frac{x(x-1)}{x^2}-\frac{(x-1)}{x^2}=\frac{(x-1)(x-1)}{x^2}$$

$=\frac{(x-1)^2}{x^2}=$ part left after 2nd draught.

$$\frac{(x-1)^2}{x^2}-\frac{1}{x}\text{ of }\frac{(x-1)^2}{x^2}=\frac{x(x-1)^2}{x^3}-\frac{(x-1)^2}{x^3}=\frac{(x-1)(x-1)^2}{x^3}$$

$=\frac{(x-1)^3}{x^3}=$ part left after 3rd draught.

$\frac{(x-1)^3}{x^3}-\frac{1}{x}$ of $\frac{(x-1)^3}{x^3}=\frac{x(x-1)^3}{x^4}-\frac{(x-1)^3}{x^4}=\frac{(x-1)(x-1)^3}{x^4}$

$=\frac{(x-1)^4}{x^4}=$ part left after 4^{th} draught

But by the question there were 81 gallons of wine left after the 4^{th} draught, or $\frac{81}{256}$ of the quantity at the beginning,

$\therefore \frac{(x-1)^4}{x^4}=\frac{81}{256}$,

$\frac{x-1}{x}=\frac{3}{4}$, by extracting the 4^{th} root.

Whence $4x-4=3x$, or $x=4$, and $\frac{1}{x}=\frac{1}{4}=$

the part of the wine drawn at each draught.

$\frac{1}{4}$ of $256=64$ gallons drawn at 1^{st} draught;

$256-64=192$, and $\frac{1}{4}$ of $192=48$ gallons at 2^{nd} draught;

$192-48=144$, and $\frac{1}{4}$ of $144=36$ gallons at 3^{rd} draught;

$144-36=108$, and $\frac{1}{4}$ of $108=27$ gallons at 4^{th} draught.

Another solution.

Let $x=$ the number of gallons of wine drawn at 1^{st} draught, and let $256=a$ for the sake of simplicity.

Then $\frac{x}{a}=$ the part of the whole wine drawn at 1^{st} draught

and $1-\frac{x}{a}=\frac{a-x}{a}=$ part left.

$\frac{x}{a}$ of $\frac{a-x}{a}=\frac{x(a-x)}{a^2}=$ part drawn at 2^{nd} draught,

and $\frac{a-x}{a}-\frac{x(a-x)}{a^2}=\frac{a(a-x)-x(a-x)}{a^2}=\frac{(a-x)(a-x)}{a^2}$

$=\frac{(a-x)^2}{a^2}=$ part left after 2^{nd} draught.

By proceeding as in the previous solution, we find

$\frac{(a-x)^4}{a^4}=$ part of the whole wine left after the 4^{th} draught.

$\therefore \frac{(a-x)^4}{a^4}=\frac{81}{256}$;

$\frac{a-x}{a}=\frac{3}{4}$, whence $x=\frac{1}{4}a=64$ from which the other draughts are easily found.

COMPLETE EQUATIONS OF THE SECOND DEGREE

Article 231.

(31) Multiplying by x and transposing, we have

$$x^2-\frac{4}{\sqrt{3}}x=-1;$$

$$x^2-\frac{4}{\sqrt{3}}x+\frac{4}{3}=-1+\frac{4}{3}=\frac{1}{3},$$ by completing the square

$$x-\frac{2}{\sqrt{3}}=\pm\frac{1}{\sqrt{3}};$$

$$x=\frac{2}{\sqrt{3}}\pm\frac{1}{\sqrt{3}}=\frac{3}{\sqrt{3}}=\sqrt{3}, \text{ or } \frac{1}{\sqrt{3}}=\frac{1}{3}\sqrt{3}.$$

(32) Multiplying both terms of the fractions in the left member by x, we have

$\frac{x^2+1}{x^2-1}+\frac{x+1}{x-1}=\frac{13}{4}$; multiplying both terms again by x^2-1,

$x^2+1+x^2+2x+1=\frac{13}{4}x^2-\frac{13}{4}$; transposing and reducing

$x^2-\frac{8}{5}x=\frac{21}{5}$.

Whence $x=3$, or $-\frac{7}{5}$.

(35) Transposing and dividing by c, we have

$x^2-\frac{2a}{c}x=-\frac{a^2-b^2}{c^2}$, completing the square

$x^2-\frac{2a}{c}x+\frac{a^2}{c^2}=\frac{a^2}{c^2}-\frac{a^2-b^2}{c^2}=\frac{b^2}{c^2}$; whence

$x-\frac{a}{c}=\pm\frac{b}{c}$; or $x=\frac{a\pm b}{c}$.

(36) Transposing ab and completing the square, we have

$$x^2-(a+b)x+\frac{(a+b)^2}{4}=-ab+\frac{(a+b)^2}{4}=\frac{(a-b)^2}{4}.$$

Whence $x=\frac{a+b}{2}\pm\frac{a-b}{2}=a$ or b.

(37) Dividing by $a-b$, transposing and completing the square.

we have $x^2-\frac{a+b}{a-b}x+\frac{(a+b)^2}{4(a-b)^2}=\frac{a^2-6ab+9b^2}{4(a-b)^2}$.

Whence $x=\frac{a+b}{2(a-b)}\pm\frac{a-3b}{2(a-b)}=1$, or $\frac{2b}{a-b}$.

(38) Transposing np, dividing by mq and completing the square

we have $x^2-\frac{mn-pq}{mq}x+\frac{(mn-pq)^2}{4m^2q^2}=\frac{(mn+pq)^2}{4m^2q^2}$.

Whence $x=\frac{mn-pq}{2mq}\pm\frac{mn+pq}{2mq}=\frac{n}{q}$, or $-\frac{p}{m}$.

(39) Observing that x^{-1} is the same as $\frac{1}{x}$, clearing of fractions by multiplying by ax, and completing the square, we have

$x^2-acx+\frac{a^2c^2}{4}=\frac{a^2c^2-4ab}{4}$.

Whence $x=\frac{ac\pm\sqrt{(a^2c^2-4ab)}}{2}$.

(40) First, $\frac{1}{(ab^2)^{-\frac{1}{2}}+(a^2b)^{-\frac{1}{2}}}=\frac{1}{\frac{1}{(ab^2)^{\frac{1}{2}}}+\frac{1}{(a^2b)^{\frac{1}{2}}}}=\frac{1}{\frac{1}{a^{\frac{1}{2}}b}+\frac{1}{ab^{\frac{1}{2}}}}$;

multiplying both terms of this fraction by ab, it becomes

$\frac{ab}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}$; the equation then becomes

$\frac{x^2}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}-(a^{\frac{1}{2}}-b^{\frac{1}{2}})x=\frac{ab}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}$.

Multiplying both members of this equation by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ it becomes

$x^2-(a-b)x=ab$.

Whence $x=a$, or $-b$

(41) Dividing both sides by $-ac$, transposing, and completing the square, we have

$x^2-\frac{ad-bc}{ac}x+\frac{(ad-bc)^2}{4a^2c^2}=\frac{(ad+bc)^2}{4a^2c^2}$.

Whence $x=\frac{ad-bc}{2ac}\pm\frac{ad+bc}{2ac}=\frac{d}{c}$, or $-\frac{b}{a}$.

(42) Square both members, and then multiply both sides by $x+12$.

(43) Square both members, omit the terms which destroy each other, transpose a, and square again; the equation will then be free from radicals, and the value of x is easily found.

(44 Multiplying both members first by $4+\sqrt{x}$ and then by $\sqrt{x}$, we have

$\sqrt{4x^2}+2\sqrt{x}=16-x$, or, since $\sqrt{4x^2}=2x$

$2\sqrt{x}=16-3x$,

$4x=256-96x+9x^2$.

Whence $x=4$, or $\frac{64}{9}$.

The first value verifies the equation when $\sqrt{x}$ is taken *plus*, and the second when it is taken *minus*.

(45) Dividing each side by the square root of x, and observing that $\sqrt{x^2}=x$, we have

$x-2=\sqrt{x}$, or

$x^2-4x+4=x$, by squaring.

Whence $x^2-5x=-4$, and $x=4$ or 1.

The first value verifies the equation when $\sqrt{x}$ is taken positively, and the second when it is taken negatively

(46) Squaring both sides we have

$x+a+x+b-2\sqrt{[x^2+(a+b)x+ab]}=2x$;

omitting $2x$ on each side, transposing $a+b$ and squaring again, we have $4x^2+4(a+b)x+4ab=(a+b)^2$; from which by transposing and reducing, we find

$$x^2+(a+b)x=\frac{(a-b)^2}{4}.$$

Whence $x^2+(a+b)x+\frac{(a+b)^2}{4}=\frac{(a+b)^2+(a-b)^2}{4}$

$=\frac{2a^2+2b^2}{4}$;

$x=-\frac{a+b}{2}\pm\frac{1}{2}\sqrt{2a^2+2b^2}$.

(47) Transposing the second term, squaring, and omitting $-2abcx$ on each side, we have

$(x^2+c^2)ab=(a^2+b^2)cx$; transposing and dividing by

$$x^2-\frac{(a^2+b^2)c}{ab}x=-c^2;$$

$$x^2-\frac{(a^2+b^2)c}{ab}x+\frac{(a^2+b^2)^2c^2}{4a^2b^2}=\frac{(a^2-b^2)c^2}{4a^2b^2};$$

$$x=\frac{(a^2+b^2)c}{2ab}\pm\frac{(a^2-b^2)c}{2ab}=\frac{ac}{b}, \text{ or } \frac{bc}{a}.$$

Another solution:

Let $\sqrt{x}=z$, then $x=z^2$; transposing $-c\sqrt{ab}$, dividing by $\sqrt{ab}$, and substituting z^2 for x, and z for $\sqrt{x}$, we have

$$z^2-\frac{(a-b)\sqrt{c}}{\sqrt{ab}}z+\frac{(a-b)^2c}{4ab}=\frac{(a-b)^2c}{4ab}+c=\frac{(a+b)^2c}{4ab}.$$

$$\text{Whence } z=\frac{(a-b)\sqrt{c}}{2\sqrt{ab}}\pm\frac{(a+b)\sqrt{c}}{2\sqrt{ab}}$$

$$=\frac{2a\sqrt{c}}{2\sqrt{ab}}=\sqrt{\frac{ac}{b}}, \text{ or } \frac{-2b\sqrt{c}}{2\sqrt{ab}}=\sqrt{\frac{bc}{a}};$$

$$\therefore \sqrt{x}=\sqrt{\frac{ac}{b}}, \text{ or } \sqrt{\frac{bc}{a}}, \text{ and } x=\frac{ac}{b}, \text{ or } \frac{bc}{a}.$$

(48) Multiplying both sides by $\sqrt{a+x}$, we have

$$a+x+\sqrt{a^2-x^2}=\frac{12a}{5},$$

$\sqrt{a^2-x^2}=\frac{7a}{5}-x$, by transposing

$a^2-x^2=\frac{49a^2}{25}-\frac{14a}{5}x+x^2$, by squaring

$x^2-\frac{7a}{5}x=-\frac{12a^2}{25}$, by reducing.

Whence $x=\frac{4a}{5}$, or $\frac{3a}{5}$.

PROBLEMS PRODUCING COMPLETE EQUATIONS OF THE SECOND DEGREE

Article 233.

(5) Let $x=$ one of the numbers, then $20-x=$ the other, and
$x(20-x)=36$.
Whence $x=2$ or 18, therefore $20-x=18$ or 2.

(6) Let $x=$ one part, then $15-x=$ the other, and
$x(15-x) : x^2+(15-x)^2 : : 2 : 5$;
$\therefore\ 4x^2-60x+450=75x-5x^2$,
reducing $x^2-15x=-50$.
Whence $x=10$ or 5, and $15-x=5$ or 10.

(7) Let $x=$ the number, then
$x(10-x)=21$.
Whence $x=7$ or 3.

(8) Let $x=$ the less part, then $24-x=$ the greater, and
$x(24-x)=35(24-x-x)$, reducing
$x^2-94x=-840$.
Whence $x=10$ or 84, the first of which is evidently only admissible; therefore, the parts are 10 and $24-10=14$.

(9) Denoting the square roots of the parts by x, and $26-x$, we have $x^2+(26-x)^2=346$,
reducing $x^2-26x=-165$.
Whence $x=15$ or 11, and $26-x=11$ or 15.

(10) Let $x=$ the square root of the number, then $x^2=$ the number, and
$x^2+x=132$.
Whence $x=11$, or -12, and $x^2=121$, or 144.

The last number is the answer to the question "What number *diminished* by its square root gives 132?"

(11) Let $x=$ the square root of the number, then $x^2=$ the number, and $x^2-x=48\frac{3}{4}$.
Whence $x=7\frac{1}{2}$, or $-6\frac{1}{2}$, and $x^2=56\frac{1}{4}$ or $42\frac{1}{4}$.

The last number is the answer to the question "What number *added* to its square root gives $48\frac{3}{4}$?

(12 Let $x=$ one of the numbers, then $41-x=$ the other, and $x^2+(41-x)^2=901$.
Whence $x=15$, or 26; and $41-x=26$, or 15.

(13) Let $x=$ the less number, then $x+8=$ the greater, and $x^2+(x+8)^2=544$.
Whence $x=12$, or -20; and $x+8=20$, or -12; hence the two numbers are 12 and 20.

(14) Let $x=$ the first cost, then $x=$ per cent. of gain, and

$$x\times\frac{x}{100}=\frac{x^2}{100}=\text{gain}.$$

$$\therefore\ x+\frac{x^2}{100}=2400.$$

Whence $x=20$.

(15) Let $x=$ the number of miles B traveled per hour, then $x+\frac{1}{4}=$ the number of miles A traveled per hour, then, also, $\frac{39}{x+\frac{1}{4}}$ and $\frac{39}{x}=$ the hours respectively which A and B traveled.

$$\therefore\ \frac{39}{x+\frac{1}{4}}+1=\frac{39}{x};$$

Whence $x=3$, and $x+\frac{1}{4}=3\frac{1}{4}$.

(16) Let $x=$ number to whom B gave, then $x+40=$ number to whom A gave, then

$\frac{1200}{x+40}=$ what A gave to each, and

$\frac{1200}{x}=$ " B " " ;

$$\therefore\ \frac{1200}{x+40}+5=\frac{1200}{x}.$$

Whence $x=80$, and $x+40=120$.

(17) Let $x=$ number of miles B traveled per day, then
$x+8=$ " " A " " , and
$\frac{1}{2}x=$ number of days each traveled;

$\therefore\ x\times\frac{1}{2}x+(x+8)\frac{1}{2}x=320$, or

$x^2+4x=320$.

Whence $x=16$, and $x+8=24$.

(18) Let $x=$ the distance in miles from C to D, then

$\frac{x}{19}=$ number of miles B traveled per day, also

$\frac{x}{19}=$ number of days B traveled, then

$32+7\times\frac{x}{19}=$ whole number of miles A traveled, and

$\frac{x}{19}\times\frac{x}{19}=\frac{x^2}{361}=$ number of miles B traveled.

$\therefore\ 32+\frac{7x}{19}+\frac{x^2}{361}=x$.

Clearing of fractions and transposing

$x^2-228x=-11552$.

Whence $x=76$, or 152.

(19) Let $x=$ the number bought, then

$\frac{240}{x}=$ number of dollars each cost, and, since $240+59$

$=299$, $\frac{299}{x-3}=$ " " " sold for;

$\therefore\ \frac{299}{x-3}-\frac{240}{x}=8$, reducing

$8x^2-83x=720$.

Whence $x=16$.

(20) Let $x=$ one of the numbers, then $100-x=$ the other;

then $x(100-x)=x^2-(100-x)^2$, reducing

$100x-x^2=-10000+200x$, or

$x^2+100x=10000$.

Whence $x=61.803+$, and $100-x=38.197$, nearly.

Or by subtracting x^2 from the square of $100-x$, and reducing we have the equation $x^2-300x=-10000$.

Whence $x=38.197$ nearly.

(21) Since each received back \$450, they both received \$900 and the whole gain was \$900—500=\$400.

Let $x=$ A's stock, then $500-x=$ B's stock.

x dollars for 5 months is the same as $5x$ dollars for 1 month.

$(500-x)$ dollars for 2 months, is the same as $2(500-x)$ $=(1000-2x)$ dollars for 1 month.

Hence the gain, \$400, is to be divided into two parts having the same ratio to each other as $5x$ and $1000-2x$. But $5x+(1000-2x)$ $=3x+1000$, therefore the parts of the gain are $\frac{5x}{3x+1000}$ and $\frac{1000-2x}{3x+1000}$, the sum of which is 1, the whole gain.

$$\therefore \text{ A's gain is } \frac{5x}{3x+1000} \text{ of } 400=\frac{2000x}{3x+1000};$$

$$\text{B's gain is } \frac{1000-2x}{3x+1000} \text{ of } 400=\frac{400000-800x}{3x+1000}.$$

But A's gain $=450-x$.

$$\therefore \frac{2000x}{3x+1000}=450-x.$$

Whence $x=200$, A's stock, and $500-x=300$, B's stock.

(22) Let $x=$ first part of 11, then $11-x=$ the second ;

also, $\frac{45}{x}=$ first part of 17, and $17-\frac{45}{x}=$ the second ;

then $(11-x)\left(17-\frac{45}{x}\right)=48$, or

$$(11-x)(17x-45)=48x.$$

Whence $x=5$, or $\frac{99}{17}$,

$11-x=6$, or $\frac{88}{17}$, and $\frac{45}{x}=9$, or $\frac{85}{11}$;

$\therefore 17-\frac{45}{x}=8$, or $\frac{102}{11}$.

Hence the numbers are 5, 6, and 9, 8,

or $\frac{99}{17}$, $\frac{88}{17}$, and $\frac{85}{11}$, $\frac{102}{11}$, either of which entirely satisfies the conditions.

(23) Let $3x=$ the first part of 21, then $x=$ the first part of 30 and $(21-3x)^2+(30-x)^2=585$;

developing and reducing

$x^2-\frac{93}{5}x=-\frac{378}{5}$.

Whence $x=6$, or $12\frac{3}{5}$, and $3x=18$, or $37\frac{4}{5}$.

Since the second value of x gives for $3x$ a number greater than 21, it is inadmissible.

The first value of x gives for the parts of 21, 18 and $21-18=3$ and for the parts of 30, 6 and $30-6=24$.

(24) Let $x=$ the first part of 19, then $19-x=$ the second part; and since the difference of the squares of the first parts of each is 72, therefore, $\sqrt{x^2+72}$ must represent the first part of 29, and $29-\sqrt{x^2+72}$ the second part.

$\therefore\ (29-\sqrt{x^2+72})^2-(19-x)^2=180,$

developing and reducing

$29\sqrt{x^2+72}=19x+186$;

squaring each side and reducing

$40x^2-589x=-2163.$

Whence $x=7$, or $\frac{309}{40}$; this gives

$\sqrt{x^2+72}=11$, or $\frac{459}{40}$.

$19-x=12$, or $\frac{451}{40}$,

$29-\sqrt{x^2+72}=18$, or $\frac{701}{40}$.

Whence the parts are 7, 12, and 11, 18,

or $\frac{309}{40}$, $\frac{451}{40}$, and $\frac{459}{40}$, $\frac{701}{40}$

Article 239a.

(1) In order that the negative answer, -9, when taken positively, shall be correct, the question should read: Required a number such, that twice its square, *diminished* by 8 times the number itself, shall be 90.

(2) From this question we see that the negative values satisfy the question equally well with the positive, the only difference being that in one case we subtract $+3$ from $+7$ and in the other, -7 from -3.

(3) Let $x=$ cost of watch, then $x=$ per cent. of loss and

$x\times\frac{x}{100}=\frac{x^2}{100}=$ actual loss.

$\therefore\ x-\frac{x^2}{100}=16$, or

$x^2-100x=-1600.$

Whence $x=50\pm30=20$, or 80, either of which fully satisfies the conditions. Thus,

$20-\frac{20}{100}$ of $20=20-4=16$;

$80-\frac{80}{100}$ of $80=80-64=16$

(4) These values of x show that no *positive* number can be found which will satisfy the question. But if the question is changed to read thus: Required a number such, that 6 times the number, diminished by the square of the number, and the result subtracted from 7, the remainder shall be 2, either of the numbers, 1 and 5, will satisfy the question.

(5) There is evidently but one solution, because if 4 is one of the numbers $10-4=6$ is the other; or, if 6 is one of the numbers $10-6=4$ is the other.

(6) These results show that the question is impossible in an arithmetical sense. This we also learn from Art. 236, since the greatest product that can be formed by dividing 10 into two parts, is 25.

(7) This question is similar to that of the problem of the lights (Art. 239), and the results may be obtained from the results there given, by making $a=a$, $b=1$, and $c=n$.

When $a=12$ and $n=4$, the parts are 8 and 4, or 24 and -12.

When $a=10$ and $n=1$, the parts are 5 and 5.

(8) Calling x the distance from the earth, $a=240000$, $b=80$ and $c=1$, we have by the solution to the problem of the lights, $x=\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$, or $x=\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$.

To prepare these formulæ for numerical calculation, multiply both terms of the first by $\sqrt{b}-\sqrt{c}$, and of the second by $\sqrt{b}+\sqrt{c}$; this gives

$$x=\frac{a(b-\sqrt{bc})}{b-c}=\frac{240000(80-\sqrt{80})}{79}=215865.5+$$

$$\text{or } x=\frac{a(b+\sqrt{bc})}{b-c}=\frac{240000(80+\sqrt{80})}{79}=270210.4+$$

$a-x=240000-215865.5=24134.5.$

and $x-a=270210.4-240000=30210.4.$

BINOMIAL SURDS.

Article 241.

(3) $A=11,\ \sqrt{B}=6\sqrt{2}=\sqrt{72}$; $C=\sqrt{121-72}=\sqrt{49}=7$;

$\therefore\ \sqrt{A+\sqrt{B}}=\sqrt{\frac{11+7}{2}}+\sqrt{\frac{11-7}{2}}=3+\sqrt{2}.$

(4) $A=7,\ \sqrt{B}=4\sqrt{3}=\sqrt{48}$; $C=\sqrt{49-48}=1$;

$\therefore\ \sqrt{A-\sqrt{B}}=\sqrt{\frac{7+1}{2}}-\sqrt{\frac{7-1}{2}}=2-\sqrt{3}.$

(5) $A=3,\ \sqrt{B}=2\sqrt{2}=\sqrt{8}$; $C=\sqrt{9-8}=1$;

$\therefore\ \sqrt{A\pm\sqrt{B}}=\sqrt{\frac{3+1}{2}}\pm\sqrt{\frac{3-1}{2}}=\sqrt{2}\pm1.$

(6) $A=13,\ \sqrt{B}=2\sqrt{30}=\sqrt{120}$; $C=\sqrt{169-120}=7$;

$\therefore\ \sqrt{A+\sqrt{B}}=\sqrt{\frac{13+7}{2}}+\sqrt{\frac{13-7}{2}}=\sqrt{10}+\sqrt{3}.$

(7) $A=17,\ \sqrt{B}=2\sqrt{60}=\sqrt{240}$; $C=\sqrt{289-240}=7$;

$\therefore\ \sqrt{A+\sqrt{B}}=\sqrt{\frac{17+7}{2}}+\sqrt{\frac{17-7}{2}}=2\sqrt{3}+\sqrt{5}.$

(8) $A=x,\ \sqrt{B}=2\sqrt{x-1}=\sqrt{4x-4}$; $C=\sqrt{x^2-4x+4}=x-2$

$\therefore\ \sqrt{A-\sqrt{B}}=\sqrt{\frac{x+x-2}{2}}-\sqrt{\frac{x-x+2}{2}}=\sqrt{x-1}-1.$

(9) $A=0,\ \sqrt{B}=2a\sqrt{-1}=\sqrt{-4a^2}$; $C=\sqrt{4a^2}=2a$;

$\therefore\ \sqrt{A+\sqrt{B}}=\sqrt{\frac{0+2a}{2}}+\sqrt{\frac{0-2a}{2}}=\sqrt{a}+\sqrt{-a}=$

$\sqrt{a}(1+\sqrt{-1}).$

(10) $A=x+y+z$, $\sqrt{B}=2\sqrt{xz+yz}=\sqrt{4xz+4yz}$.

$C=\sqrt{A^2-B}=\sqrt{(x^2+y^2+z^2+2xy+2xz+2yz-4xz-4yz)}$
$=\sqrt{(x^2+y^2+z^2+2xy-2xz-2yz)}=x+y-z.$

$$\therefore \quad \sqrt{A+\sqrt{B}}=\sqrt{\frac{x+y+z+x+y-z}{2}}+\sqrt{\frac{x+y+z-(x+y-z)}{2}}$$
$$=\sqrt{x+y}+\sqrt{z}.$$

(11) To find the square root of $bc+2b\sqrt{bc-b^2}$.

$A=bc$, $\sqrt{B}=2b\sqrt{bc-b^2}=\sqrt{4b^3c-4b^4}$;

$\sqrt{A^2-B}=\sqrt{b^2c^2-4b^3c+4b^4}=bc-2b^2=C.$

$$\sqrt{A+\sqrt{B}}=\sqrt{\frac{bc+bc-2b^2}{2}}+\sqrt{\frac{bc-bc+2b^2}{2}}$$
$$=\sqrt{bc-b^2}+b.$$

To find the square root of $bc-2b\sqrt{bc-b^2}$.

$A=bc$, $\sqrt{B}=2b\sqrt{bc-b^2}=\sqrt{4b^3c-4b^4}$;

$\sqrt{A^2-B}=\sqrt{b^2c^2-4b^3c+4b^4}=bc-2b^2=C.$

$$\sqrt{A-\sqrt{B}}=\sqrt{\frac{bc+bc-2b^2}{2}}-\sqrt{\frac{bc-bc+2b^2}{2}}$$
$$=\sqrt{bc-b^2}-b.$$

$$\therefore \quad \sqrt{bc+2b\sqrt{bc-b^2}}=\pm\ \{\sqrt{bc-b^2}+b\}$$
$$\sqrt{bc-2b\sqrt{bc-b^2}}=\pm\ \{\sqrt{bc-b^2}-b\}$$

Sum $=\pm 2\sqrt{bc-b^2}$. *Ans.*

To verify this result with numbers, let $b=1$ and $c=26$,

then $\sqrt{bc+2b\sqrt{bc-b^2}}=\sqrt{26+2\sqrt{25}}=\pm\sqrt{26+10}=\pm 6$

$\sqrt{bc-2b\sqrt{bc-b^2}}=\sqrt{26-2\sqrt{25}}=\pm\sqrt{26-10}=\pm 4$

and $\pm 6\pm 4=\pm 10$.

But $\pm 2\sqrt{bc-b^2}=\pm 2\sqrt{25}=\pm 10$.

The preceding example may be found in the French edition of Bourdon's Algebra, where the answer is given $+2b$ which may be obtained by giving the sign $\pm$ to b.

TRINOMIAL EQUATIONS.

Article 242.

(5) $x^4-25x^2=-144,$

$x^4-25x^2+\frac{625}{4}=+\frac{625}{4}-144=\frac{49}{4};$

$x^2-\frac{25}{2}=\pm\frac{7}{2},$

$x^2=\frac{25}{2}\pm\frac{7}{2}=16$ or 9.

$x=\pm4$ or ±3.

(6) $5x^4+7x^2=6732$, divide by 5 and complete the square,

$x^4+\frac{7}{5}x^2+\frac{49}{100}=\frac{49}{100}+\frac{6732}{5}=\frac{134689}{100},$

$x^2+\frac{7}{10}=\pm\frac{367}{10},$

$x^2=-\frac{7}{10}\pm\frac{367}{10}=36$, or $-\frac{374}{10}=-\frac{187}{5};$

$x=\pm6$, or $\pm\sqrt{-\frac{187}{5}}=\pm\sqrt{\frac{-935}{25}}=\pm\frac{1}{5}\sqrt{-935}.$

The last answer in the book is $\pm\frac{1}{10}\sqrt{-3740}$, which is the same as $\pm\frac{1}{5}\sqrt{-935}$, since $\sqrt{-3740}=\sqrt{-935\times4}=2\sqrt{-935}$.

(7) $9x^6-11x^3=488$, divide by 9 and complete the square,

$x^6-\frac{11}{9}x^3+\frac{121}{324}=\frac{121}{324}+\frac{488}{9}=\frac{17689}{324};$

$x^3-\frac{11}{18}=\pm\frac{133}{18},$

$x^3=\frac{11}{18}\pm\frac{133}{18}=6$, or $-\frac{183}{27},$

$x^3=8$, or $-\frac{183}{27},$

$x=\pm2$, or $\pm\sqrt[3]{-\frac{1}{27}\times183}=\pm\frac{1}{3}\sqrt[3]{-183}$, or $\pm\frac{1}{3}\sqrt[3]{183}$

(8) Completing the square by adding $\frac{1}{4}$ to each member, we have

$x^3-x^{\frac{3}{2}}+\frac{1}{4}=\frac{62001}{4};$

$x^{\frac{3}{2}}-\frac{1}{2}=\pm\frac{249}{2};$

$x^{\frac{3}{2}}=\frac{1}{2}\pm\frac{249}{2}=125$, or $-\mathbf{124};$

$x^3=(125)^2=(5\times5\times5)^2$, or $(-124)^2;$

$x=5^2=25$, or $(-124)^{\frac{2}{3}}.$

(9) Arranging the terms and completing the square, by adding $\frac{1}{4}$ to each member, we have

$x^{\frac{5}{3}}+x^{\frac{5}{6}}+\frac{1}{4}=\frac{4225}{4}$;

$x^{\frac{5}{6}}+\frac{1}{2}=\pm\frac{65}{2}$

$x^{\frac{5}{6}}=-\frac{1}{2}\pm\frac{65}{2}=32$, or -33;

$x^{\frac{1}{6}}=2$, or $(-33)^{\frac{1}{5}}$,

$x=64$, or $(-33)^{\frac{6}{5}}$.

(10) Let $\sqrt{x+5}=y$, then $x+5=y^2$, and the equation becomes, by substitution and transposition,

$y^2-y=6$.

Whence $y=3$, or -2.

$\therefore \sqrt{x+5}=3$, or -2,

$x+5=9$, or 4,

$x=4$, or -1.

(11) Add 3 to each side, let $\sqrt{x^2-3x+11}=y$, then by substituting the value of y^2 and transposing, we have

$y^2-2y=3$.

Whence $y=3$, or -1.

$\therefore \sqrt{x^2-3x+11}=9$ or 1.

From which, by squaring and solving the resulting equations, we readily find $x=2$, or 1, or $\frac{3}{2}\pm\frac{1}{2}\sqrt{-31}$.

(12) Add 18 to each member, let $\sqrt{x^2-7x+18}=y$, then $y^2+y=42$.

Whence y, or $\sqrt{x^2-7x+18}=6$, or -7.

Squaring, $x^2-7x+18=36$, or 49.

Whence $x=9$, -2, or $\frac{1}{2}(7\pm\sqrt{173})$.

(13) To render this equation of a quadratic form, the quantity in the parenthesis in the right member must be made the same as that in the parenthesis on the left. This may be done by adding -7 to the quantity in the vinculum, and its equal, $+7\times11=77$, without; the equation then becomes

$(x^2-9)^2=3+77+11(x^2-2-7)=80+11(x^2-9)$.

Putting y and y^2 to represent x^2-9, and $(x^2-9)^2$, we have
$y^2-11y=80$;
whence $y=16$, or -5.
$\therefore\ x^2-9=16$, or -5, and $x=\pm5$, or ±2.

(14) Transposing $\frac{8}{x}$ and putting $y=x+\frac{8}{x}$, we have
$y^2+y=42$.
Whence $y=+6$, or -7;
$\therefore\ x+\frac{8}{x}=+6$, or -7,
and $x^2-6x=-8$, or $x^2+7x=-8$.
Whence $x=4$ or 2, or $\frac{1}{2}(-7\pm\sqrt{17})$.

(15) This equation may be placed under the form
$$x^4\left(1+\frac{1}{3x}\right)^2-3x^2\left(1+\frac{1}{3x}\right)=70.$$
Putting $x^2\left(1+\frac{1}{3x}\right)=y$, we have
$y^2-3y=70$;
whence $y=10$, or -7.
$\therefore\ x^2\left(1+\frac{1}{3x}\right)=10$, or -7,
$x^2+\frac{1}{3}x=10$, or -7.
Whence $x=3$, or $-3\frac{1}{3}$, or $\frac{1}{6}(-1\pm\sqrt{-251})$

(16) Multiplying both sides by $\sqrt{x}$, we have
$x\sqrt{6-x^2}=1+x^2$,
Squaring $x^2(6-x^2)=1+2x^2+x^4$,
transposing and reducing $x^4-2x^2=-\frac{1}{2}$.
Whence $x=\pm\sqrt{(1\pm\frac{1}{2}\sqrt{2})}$.

Article 243.

(2) We find the square root is x^2-x, with the remainder $-3x^2+3x$; hence the equation may be written thus,
$(x^2-x)^2-3(x^2-x)=108$.

Putting $x^2-x=y$, we find $y=12$, or -9.

$\therefore x^2-x=12$, or -9.

Whence $x=4$, or -3, or $\frac{1}{2}(1\pm\sqrt{-35})$.

(3) The square root of the left member is x^2-x, with the remainder $-x^2+x$; hence the equation may be written thus, $(x^2-x)^2-(x^2-x)=30$.

Putting $x^2-x=y$, we find $y=6$, or -5.

$\therefore x^2-x=6$, or -5.

Whence $x=3$, or -2, or $\frac{1}{2}(1\pm\sqrt{-19})$.

(4) Multiplying both sides by x, we then find the square root of the left member is x^2-3x with the remainder $+2x^2-6x$; hence, the equation may be written thus,

$(x^2-3x)^2+2(x^2-3x)=0$.

Let $x^2-3x=y$, then $y^2+2y=0$, and $y=0$, or -2.

$\therefore x^2-3x=0$, or -2.

Whence $x=0$, or 3, or 2, or 1.

The value, $x=0$, does not satisfy the given equation, but is a root of the equation $x(x^3-6x^2+11x-6)=0$, and was introduced by multiplying the given equation by x.

(5) The square root of the left member is x^2-3x with the remainder $-4x^2+12x$; hence the equation may be written thus, $(x^2-3x)^2-4(x^2-3x)=60$.

Let $x^2-3x=y$, then $y^2-4y=60$;

whence $y=10$, or -6.

$\therefore x^2-3x=10$, or -6.

Whence $x=5$, -2, or $\frac{1}{2}(3\pm\sqrt{-15})$.

(6) The square root of the left member is x^2-4x, with the remainder $-6x^2+24x$; hence, the equation may be written $(x^2-4x)^2-6(x^2-4x)=-5$.

Let $x^2-4x=y$, then $y^2-6y=-5$; from which $y=5$ or 1;

$\therefore x^2-4x=5$ or 1.

Whence $x=5$, or -1, or $2\pm\sqrt{5}$.

(7) Multiplying both members by 4 to clear the equation of fractions and render the first term a perfect square; then transposing $16x^3$, we find the square root of the left mem

ber is $4x^2-2x$, with the remainder $-4x^2+2x$; hence the equation may be written

$(4x^2-2x)^2-(4x^2-2x)=132.$

Let $4x^2-2x=y$, then $y^2-y=132$, and $y=12$, or -11.

$\therefore\ 4x^2-2x=12$, or -11.

Whence $x=2, -\frac{3}{2}$, or $\frac{1}{4}(1\pm\sqrt{-43})$.

(8) Observe that $\dfrac{12+\frac{1}{2}x}{3x}=\dfrac{4}{x}+\dfrac{1}{6}$, then omitting $\dfrac{1}{6}$ on each side, and multiplying both sides of the equation by $14x^3$ to clear it of fractions; after transposing we nave

$x^4-14x^3+56x^2-49x=60.$

The square root of the left member is x^2-7x with the remainder $7x^2-49x$; hence, the equation may be written

$(x^2-7x)^2+7(x^2-7x)=60.$

Let $x^2-7x=y$, then $y^2+7y=60$, from which we find $y=5$, or -12.

$\therefore\ x^2-7x=5$, or -12.

Whence $x=4$, or 3, or $\frac{1}{2}(7\pm\sqrt{69})$.

SIMULTANEOUS EQUATIONS OF THE SECOND DEGREE CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

Article 245.

NOTE.— Instead of indicating each step of the solution of the examples in this article, it has only been deemed necessary in most cases to point out the particular step on which the solution depends.

(5) Subtract the square of the first equation from the second, then add the remainder to the second, and extract the square root, which will give $x+y$.

(6) Add twice the second equation to the first and extract the square root; also, subtract twice the second equation from the first and extract the square root.

(7) Subtract the second equation from the square of the first; then subtract the remainder from the second equation and extract the square root.

(8) Divide the first equation by the second, this will give $x+y$.

(9) From the cube of the first equation subtract the second, divide the remainder by 3, and we have $xy(x+y)=308$; divide by $x+y=11$, and we have $xy=28$. Having $x+y$ and xy, we can readily find x and y, as in Form 1, Art. 245.

Or, thus, Divide the second equation by the first, subtract the quotient from the square of the first, and divide by 3, which will give xy.

(10) From the first equation by transposing and extracting the cube root of both members, we have $x=2y$; then by substitution in the second we readily find the value of y, and then x.

(11) Subtract the second equation from the first; add the remainder to the first and extract the square root, which will give $x+y=\pm12$, then divide the second equation by this, and we have $x-y=\pm2$.

(12) Divide the first equation by the second, this gives $x+y=8$; from the square of this subtract the second equation, and divide by 3, this gives $xy=15$; subtract this from the second equation, and extract the square root, which will give $x-y=\pm2$.

(13) Subtract the first equation from the square of the second, this gives $xy=48$; subtract three times this equation from the first and extract the square root, this gives $x-y=\pm8$.

(14) Divide the first equation by the second, transpose $3\frac{1}{2}xy$, and subtract the resulting equation from the square of the second, this gives $\frac{1}{2}xy=4$, or $xy=8$; then by the method explained in Form 1, Art. 245, we readily find $x+y=\pm6$.

(15) Dividing the first equation by the second, we find $x^2-xy+y^2=7$; subtracting this from the second equation we have $2xy=6$, or $xy=3$; then adding this to the second equation and extracting the square root, we find $x+y=\pm4$; also, subtracting $xy=3$ from the equation $x^2-xy+y^2=7$, and extracting the square root, we find $x-y=\pm2$

(16) Let $x^{\frac{1}{2}}=P$ and $y^{\frac{1}{2}}=Q$, the equations then become

$P^2-Q^2=P+Q$, (1)

$P^3-Q^3=37$. (2)

Dividing each side of (1) by $P+Q$, we have $P-Q=1$, then dividing each side of (2) by $P-Q=1$, we have

$P^2+PQ+Q^2=37$, (3)

$P^2-2PQ+Q^2=1$, by squaring $P-Q=1$.

Subtracting and dividing by 3, we find $PQ=12$, then by adding this to (3) and extracting the square root, we find

$P+Q=\pm 7$,

but $P-Q=1$.

Whence $P=4$, or -3, and $Q=3$, or -4,

$\therefore x=16$, or 9, and $y=9$, or 16.

(17) Let $x^{\frac{1}{4}}=P$ and $y^{\frac{1}{3}}=Q$, then by substitution the equations become $P+Q=5$,

$P^2+Q^2=13$.

The values of P and Q found as in example 1, page 212 are $P=2$ or 3, $Q=3$ or 2.

$\therefore x^{\frac{1}{4}}=2$ or 3, and $x=16$ or 81;

$y^{\frac{1}{3}}=3$ or 2, and $y=27$ or 8.

(18) Let $x^{\frac{1}{3}}=P$ and $y^{\frac{1}{3}}=Q$, then by substitution the equations become $P+Q=5$,

$P^3+Q^3=35$.

The values of P and Q found as in example 3, page 213, are $P=2$ or 3, $Q=3$ or 2.

$\therefore x^{\frac{1}{3}}=2$ or 3, and $x=8$ or 27;

$y^{\frac{1}{3}}=3$ or 2, and $y=27$ or 8.

(19) Let $x^{\frac{1}{2}}=P$ and $y^{\frac{1}{2}}=Q$, then by substitution the equations become $P+Q=4$,

$P^3+Q^3=28$.

The values of P and Q found as in the preceding example, are $P=3$ or 1, $Q=1$ or 3.

$\therefore\ x^{\frac{1}{2}}=3$ or 1, and $x=9$ or 1.

$y^{\frac{1}{2}}=1$ or 3, and $y=1$ or 9.

(20) Square both members of the first equation, and from the result subtract four times the cube of the second, and we have $x^6-2x^3y^3+y^6=112225.$
extracting the sq. root $x^3-y^3=\pm 335$;
but $x^3+y^3=351.$

Whence, by adding and subtracting, dividing by 2 and extracting the cube root, we have $x=7$ or 2, and $y=2$ or 7.

(21) Raising both sides of (1) to the fourth power, we have

$$x^4+4x^3y+6x^2y^2+4xy^3+y^4=256,$$
$$\text{but } x^4+y^4=82;$$
$$\therefore\ 2x^4+4x^3y+6x^2y^2+4xy^3+2y^4=338;$$
$$\text{or}\quad x^4+2x^3y+3x^2y^2+2xy^3+y^4=169.$$

Extracting the sq. root $x^2+xy+y^2=13$ (3).

Squaring eq. (1) $x^2+2xy+y^2=16$;

Subtracting $xy=3$; hence $3xy=9$, and subtracting this from (3), and extracting the square root of the resulting equation, we get $x-y=\pm 2$; from this, and $x+y=4$, we get $x=3$, or 1, and $y=1$, or 3.

(22) Adding the three equations together, and dividing by 2, we have $xy+xz+yz=\dfrac{a+b+c}{2}$, (4).

Subtracting from this successively the three given equations, we have $yz=\dfrac{b+c-a}{2}$, (5)

$$xz=\frac{a+c-b}{2}, \qquad (6)$$

$$xy=\frac{a+b-c}{2}. \qquad (7)$$

Multiplying the three equations together, and extracting the square root, $xyz=\sqrt{\left\{\dfrac{(a+b-c)(a+c-b)(b+c-a)}{8}\right\}}$. (8)

Dividing eq (8) by equations (5), (6), (7), respectively, we obtain the values of x, y, and z. Thus, to find x

$$\frac{xyz}{yz}=x=\frac{\sqrt{\left\{\frac{(a+b-c)(a+c-b)(b+c-a)}{8}\right\}}}{\frac{b+c-a}{2}}$$

$$=\sqrt{\left\{\frac{(a+b-c)(a+c-b)(b+c-a)}{8}\times\frac{4}{(b+c-a)^2}\right\}}.$$

$$=\pm\sqrt{\frac{(a+b-c)(a+c-b)}{2(b+c-a)}}.$$

ADFECTED EQUATIONS.

Article 250.

(3) Adding the two equations together, and dividing by 2, we find $x^2+x=240$.

Subtracting the second equation from the first, and dividing by 2, we find $y^2+y=90$.

(4) Multiplying the first equation by 4, and subtracting the result from the second, and transposing, we find

$y^2-18y=45$.

Whence $y=3$, or 15, and $x=14-4y=2$, or -46.

(5) From the first equation $y=\frac{3x+14}{2}=1\frac{1}{2}x+7$, this substituted in the second gives

$3x^2+2(1\frac{1}{2}x-4)^2=14$,

developing and reducing, we have

$x^2-\frac{16}{5}x=-\frac{12}{5}$.

Whence $x=2$, or $1\frac{1}{5}$, and $y=10$, or $8\frac{4}{5}$.

(6) Clearing the second equation of fractions by multiplying by xy, and substituting the value of $x=y+2$, found from the first equation, and reducing, we have

$y^2-\frac{7}{4}y=\frac{15}{4}$, and $y=3$, or $-1\frac{1}{4}$.

(7) Let $y=tx$, then substituting this instead of y, finding the value of x^2 from the resulting equations, placing these values equal to each other, and reducing, we find $t=\pm1\frac{1}{2}$.

Then substituting this in the value of x^2, we find x and thence y.

(8) Let $y=tx$, then substituting this instead of y, finding the value of x^2 from the resulting equations, placing these values equal to each other, and reducing, we find $t=+1\frac{1}{2}$, or $-\frac{4}{5}$. Having this, the values of x and y are readily found by substitution.

(9) Let $xy=v$, then by substituting in the first equation, and transposing, we have $v^2+4v=96$, from which we find v, or $xy=+8$, or -12. Having the values of $x+y$ and xy, we readily find x and y by the method explained in Form 1, Art. 245.

(10) Let $\frac{x}{y}=v$, then by substituting the values of v and v^2 in the first equation we find $v^2+4v=\frac{85}{9}$, from which v or $\frac{x}{y}=+\frac{5}{3}$, or $-\frac{17}{3}$. Then, from these equations, and $x-y=2$, we readily find x and y.

(11) Denoting xy by v, the first equation becomes, by substitution and transposition, $v^2+8v=180$, from which we find v or $xy=+10$, or -18.

From the equations $xy=+10$, and $x+3y=11$, we find $x=5$, or 6, and $y=2$, or $\frac{5}{3}$.

From the equations $xy=-18$, and $x+3y=11$, we find $x=\frac{11}{2}\mp\frac{1}{2}\sqrt{337}$, and $y=\frac{11}{6}\pm\frac{1}{6}\sqrt{337}$.

(12) Let $\sqrt{x+y}=v$, then from the 1st equation, we have $v^2+v=12$, from which $v=3$, or -4; hence, v^2, or $x+y=9$, or 16. Having the values of $x+y$ and x^2+y^2, we can find the values of x and y by the method explained in example 1st, Art. 245. The equations $x+y=9$, and $x^2+y^2=41$, give $x=5$ or 4, and $y=4$ or 5. The equations $x+y=16$, and $x^2+y^2=41$, give

$$x=8\pm\tfrac{1}{2}\sqrt{-174}, \text{ and } y=8\mp\tfrac{1}{2}\sqrt{-174}.$$

(13) Adding twice the second equation to the first, we have

$$x^2+2xy+y^2+x+y=30,$$

or $(x+y)^2+(x+y)=30$.

Whence $x+y=+5$, or -6.

From these equations and the value of $xy=6$, we readily find the values of x and y.

(14) Transposing $2xy$ in the first equation, and then adding both equations together, we have

$x^2+2xy+y^2+4x+4y=117$,

or $(x+y)^2+4(x+y)=117$.

Whence $x+y=+9$, or -13, and $x=9-y$, or $-13-y$.

Substituting these values of x in the second of the given equations, we have $y^2+2y=35$, or $y^2+2y=57$.

From the 1st equation we find $y=5$, or -7; whence $x=9$ or 16. From the 2nd equation we find $y=-1\pm\sqrt{58}$; whence $x=-12\pm\sqrt{58}$.

(15) Let $\frac{1}{x}=v$ and $\frac{1}{y}=z$, the equations then become

$v+z=a$, (1)

$v^2+z^2=b$. (2)

Subtracting the second equation from the square of the first and then subtracting the remainder from the second, and extracting the square root, we find v or $\frac{1}{x}=\frac{a\pm\sqrt{2b-a^2}}{2}$, and z or $\frac{1}{y}=\frac{a\pm\sqrt{2b-a^2}}{2}$; whence $x=\frac{2}{a\pm\sqrt{2b-a^2}}$, and $y=\frac{2}{a\pm\sqrt{2b-a^2}}$.

(16) From the first equation $x=\frac{12}{y+y^2}$, and from the second $x=\frac{18}{1+y^3}$; whence $\frac{12}{y+y^2}=\frac{18}{1+y^3}$, or $12(1+y^3)=18(y+y^2)$; dividing each member by 6, and then by $1+y$, we have $2(1-y+y^2)=3y$. From this equation we readily find $y=2$, or $\frac{1}{2}$; whence $x=2$, or 16.

(17) Adding twice the second equation to the first, we have

$x^2+2xy+y^2+x+y=156$,

or $(x+y)^2+(x+y)=156$.

Whence $x+y=+12$, or -13.

By substituting the value of $x+y$ in the second equation, we find $xy=27$, or 52.

From the equations $x+y=12$, and $xy=27$, we find $x=9$ or 3 and $y=3$, or 9.

From the equations $x+y=-13$, and $xy=52$, we find
$x=\frac{1}{2}(-13\pm\sqrt{-39})$, and $y=\frac{1}{2}(-13\mp\sqrt{-39})$.

18) From the 1st equation, by transposing, we find
$(x+y)^2-3(x+y)=28$.
Whence $x+y=7$, or -4, and $y=7-x$, or $-4-x$.

Substituting the value of y instead of y in the second equation, we have $2x^2-17x=35$, or $2x^2+5x=-35$.

From the first of these equations we find $x=5$, or $3\frac{1}{2}$, whence $x=2$, or $3\frac{1}{2}$.

From the second we find $x=-\frac{5}{4}\pm\frac{1}{4}\sqrt{-255}$, and $y=-\frac{11}{4}\mp\frac{1}{4}\sqrt{-255}$.

(19) Let $\left(\frac{3x}{x+y}\right)^{\frac{1}{2}}=v$, then $\left(\frac{x+y}{3x}\right)^{\frac{1}{2}}=\frac{1}{v}$, and $v+\frac{1}{v}=2$, or
$v^2-2v=-1$, whence $v=+1$.
$\therefore \frac{3x}{x+y}=1$, whence $2x=y$.

Substituting $2x$ instead of y in the second equation we have $2x^2-3x=54$.

Whence $x=6$, or $-4\frac{1}{2}$; hence $y=12$, or -9.

(20) Transposing $3y$ and adding 5 to each member, we have
$x^2+3y+5+4(x^2+3y+5)^{\frac{1}{2}}=60$.
Let $(x^2+3y+5)^{\frac{1}{2}}=v$, then
$v^2+4v=60$.
Whence $v=6$, or -10.
$\therefore x^2+3y+5=36$, or 100.

Finding the value of y from the second equation and substituting it in the preceding equations, and reducing, we have $x^2+\frac{18}{7}x=\frac{265}{7}$, or $x^2+\frac{18}{7}x=\frac{713}{7}$.

From the first equation $x=5$, or $-\frac{53}{7}$, then since $y=\frac{6x-16}{7}$, we find $y=2$, or $-\frac{430}{49}$.

From the second equation we find $x=-\frac{9}{7}\pm\frac{4}{7}\sqrt{317}$, and thence $y=-\frac{166}{49}\pm\frac{24}{49}\sqrt{317}$

(21) The first equation is

$$\frac{y}{(x+y)^{\frac{3}{2}}}+\frac{\sqrt{x+y}}{y}=\frac{17}{4\sqrt{x+y}}.$$

Multiplying both members by $\sqrt{x+y}$, we have

$$\frac{y}{x+y}+\frac{x+y}{y}=\frac{17}{4}.$$

Let $\frac{y}{x+y}=v$, then $v+\frac{1}{v}=\frac{17}{4}$.

Whence v or $\frac{y}{x+y}=4$, or $\frac{1}{4}$.

From the equations $\frac{y}{x+y}=4$, or $\frac{1}{4}$, we find $x=-\frac{3}{4}y$, or $+3y$.

Substituting the first value of x in the equation $x=y^2+2$, we find $y=-\frac{3}{8}\pm\frac{1}{8}\sqrt{-119}$; hence $x=\frac{9}{32}\mp\frac{3}{32}\sqrt{-119}$.

Substituting the second value of x in the equation $x=y^2+2$, we find $y=2$ or 1; hence $x=6$ or 3.

QUESTIONS PRODUCING SIMULTANEOUS EQUATIONS OF THE SECOND DEGREE CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

Article 251.

NOTE.— As the first five examples may be solved without completing the square, their solutions will be given in this form.

(1) Let x represent the greater number and y the less,

then $y(x+y)=4x$, (1)

and $x(x+y)=9y$. (2)

Multiplying the equations together, dividing both members by xy, and extracting the square root, we find $x+y=6$.

Substituting 6 for $x+y$ in (1), we have $6y=4x$, or $y=\frac{2}{3}x$. Then from the equation $x+y=6$, we have $x+\frac{2x}{3}=x$; whence $x=3.6$, and $y=2.4$.

(2) Let $x=$ the digit in ten's place and $y=$ the digit in unit's place, then $10x+y=$ the number, and

$x(10x+y)=10x^2+xy=46$, (1)

also $x(x+y)=\quad x^2+xy=10$ (2)

Subtracting (2) from (1), $9x^2=36$, whence $x=2$, and y is readily found $=3$.

(4) Let $x=$ the greater number and y the less, then

$$(x-y)(x^2-y^2)=32,$$
$$(x+y)(x^2+y^2)=272.$$

For the method of finding the values of x and y, see the Algebra, example 4, page 213.

(5) Let x and y represent the numbers, then

$$xy=10,$$
$$x^3+y^3=133.$$

For the method of finding the values of x and y, see the solution to example 20, Art. 245, (page 148, of Key).

(6) Let $x=$ the greater number and y the less, then

$$\frac{(x+y)x}{y}=24, \qquad (1)$$
$$\frac{(x+y)y}{x}=6. \qquad (2)$$

Dividing the first equation by the second, we have

$$\frac{(x+y)x}{y}\times\frac{x}{(x+y)y}=\frac{x^2}{y^2}=\frac{24}{6}=4.$$

Whence $\frac{x}{y}=2$, and $x=2y$; substituting this value of x in (1) we find $6y=24$, or $y=4$; hence $x=8$.

(6) Let $x=$ the less number, then $x+15=$ the greater, and

$$\frac{x(x+15)}{2}=x^3.$$

Dividing by x, and completing the square, we find $x=3$, or $-2\frac{1}{2}$; hence $x+15=18$, or $12\frac{1}{2}$, therefore the numbers are 3 and 13, or $-2\frac{1}{2}$, and $12\frac{1}{2}$.

(7) Let $x=$ the greater number and $y=$ the less, then

$$xy=24,$$
$$\text{and } x^2-y^2=20.$$

From the first equation $y=\frac{24}{x}$; substituting this for y in the second equation, and clearing of fractions, we find $x^4-20x^2=576$.

Whence $x^2=36$, and $x=6$; hence $y=4$.

(8) Let $x=$ the greater number, and $y=$ the less, then

$x^2+xy=120,$ (1)

$xy-y^2=16.$ (2)

Let $y=vx$, then by substitution the equations become

$$x^2+x^2v=120, \text{ whence } x^2=\frac{120}{1+v};$$

and $vx^2-v^2x^2=16$, whence $x^2=\dfrac{16}{v-v^2}$.

$$\frac{120}{1+v}=\frac{16}{v-v^2}.$$

From this equation we find $v=\frac{2}{3}$, or $\frac{1}{5}$,

then $x^2=\dfrac{120}{1+v}=72$, or 100,

and $x=6\sqrt{2}$, or 10,

$y=4\sqrt{2}$, or 2.

The answers 10 and 2 are the only ones given in the Algebra, but it may be easily shown that the others are strictly true in an arithmetical sense.

(9) Let x and y represent the numbers, then

$x^2+y^2+x+y=42,$ (1)

$xy=15.$ (2)

If we add twice the second equation to the first, the resulting equation is

$(x+y)^2+(x+y)=72.$

Whence $x+y=8$, or -9.

Having $x+y$ and xy, the values of x and y are to be found as in example 13, Art. 250. (See Key, page 150.)

We thus find $x=5$ or 3, and $y=3$ or 5,

or $x=\dfrac{-9\pm\sqrt{21}}{2}$, and $y=\dfrac{-9\mp\sqrt{21}}{2}$.

(10) Let x and y represent the numbers, then

$x+y+xy=47,$ (1)

$x^2+y^2-(x+y)=62.$ (2)

For the method of finding the values of x and y see the solution to example 17, Art. 250, (Key, page 151).

(11) Let $x=$ the greater number, and $y=$ the less, then

$$xy=x+y, \qquad (1)$$
$$x^2-y^2=x+y. \qquad (2)$$

Dividing each member of (2) by $x+y$, we have

$x-y=1$, or $x=y+1$.

Substituting this value of x in (1), and reducing, we find

$y^2-y=1$.

Whence $y=\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$, and $x=\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$.

In order that the numbers may be positive, we can only use the upper sign; this gives $x=2.668$, and $y=1.668$ nearly.

12) Let $x=$ the less number, and $xy=$ the greater, then

$$x^2y=x^2y^2-x^2, \qquad (1)$$
$$x^2y^2+x^2=x^3y^3-x^3. \qquad (2)$$

Dividing each member of (1) by x^2, we have $y=y^2-1$, or $y^2=y+1$, from which we find $y=\frac{1}{2}+\frac{1}{2}\sqrt{5}$.

Dividing each member of (2) by x^2, we have $y^2+1=x(y^3-1)$, but $y^2=y+1$, and multiplying both sides by y, we have $y^3=y^2+y=y+y+1=2y+1$. Substituting these values of y^2 and y^3, the equation becomes

$$y+2=x(2y);$$

$$\text{hence } x=\frac{y+2}{2y}=\frac{\frac{5}{2}+\frac{1}{2}\sqrt{5}}{1+\sqrt{5}}=\frac{(\frac{5}{2}+\frac{1}{2}\sqrt{5})}{1+\sqrt{5}}\times\frac{1-\sqrt{5}}{1-\sqrt{5}}$$

$$=\frac{\frac{5}{2}-2\sqrt{5}-\frac{5}{2}}{1-5}=\frac{-2\sqrt{5}}{-4}=\frac{1}{2}\sqrt{5}.$$

$\therefore\ x=\frac{1}{2}\sqrt{5}$, and $xy=(\frac{1}{2}+\frac{1}{2}\sqrt{5})(\frac{1}{2}\sqrt{5})=\frac{1}{4}\sqrt{5}+\frac{5}{4}$

$=\frac{1}{4}(5+\sqrt{5})$.

(13) Let $x=$ the price in dollars of a pound of mace, and $y=$ the price of a pound of cloves, then

$$80x+100y=65; \qquad (1)$$

$\frac{20}{y}=$ pounds of cloves for 20 dollars, and

$\frac{10}{x}=$ " " mace for 10 dollars.

$$\therefore\ \frac{20}{y}-60=\frac{10}{x}. \qquad (2)$$

From these equations we readily find $x=\frac{1}{2}$ dollar $=50$ cts. and $y=\frac{1}{4}$ dollar $=25$ cts.

(14) This question may be solved by using only one unknown quantity. Thus,

Let $3x=$ A's gain, then $20x=$ B's stock, and $100-3x=$ B's gain, and $40x-200=$ A's stock.

$\therefore\ 40x-200 : 20x :: 3x : 100-3x.$

Since the product of the means is equal to the product of the extremes, $60x^2=(40x-200)(100-3x)$;

reducing $x^2-\frac{230}{9}x=-\frac{1000}{9}$.

Whence $x=20$, hence $3x=60=$ A's gain, &c.

(15) Let x and y represent the numbers, then by the question

$$xy+x+y=23, \qquad (1)$$
$$x^2+y^2-5(x+y)=8. \qquad (2)$$

Adding twice eq. (1) to eq. (2), we have

$$x^2+2xy+y^2-3(x+y)=54,$$
or $$(x+y)^2-3(x+y)=54.$$

This is a quadratic form and we readily find $x+y=9$, then by substituting the value of $x+y$ in eq. (1) we find $xy=14$. Having $x+y$ and xy, we can find x and y. (See Form 1, Art. 245.)

(16) Let x, y, and z represent the numbers, then

$$x-y-(y-z)=x-2y+z=5, \qquad (1)$$
$$x+y+z=44, \qquad (2)$$
$$xyz=1950. \qquad (3)$$

Subtracting eq. (1) from (2), and dividing by 3, we find $y=13$; then substituting this value of y in (2) and (3), we have

$$x+z=31,$$
$$xz=150.$$

Whence by Form 1st, Art. 245, we readily find $x=25$, and $z=6$.

(17) Let x, y, and z represent the parts, then

$$x+y+z=26, \qquad (1)$$
$$x^2-y^2=y^2-z^2, \qquad (2)$$
$$x^2+y^2+z^2=300. \qquad (3)$$

From eq. (2) by transposing y^2-z^2, we have
$x^2-2y^2+z^2=0$.
Subtracting this from (3), dividing by 3, and extracting the square root, we have $y=10$; then by substitution, equations (1) and (2) reduce to $x+y=16$,
and $x^2+y^2=200$.

These equations are similar to those in example 1, Art. 245, and may be solved in a similar manner.

(18) Let x and y represent the number of men respectively in the fronts of the columns A and B, when each consisted of as many ranks as it had men in front; then x^2 and y^2 represent the number of men in the respective columns.

$\therefore\ \frac{x^2}{y}=$ number of men in rank, when A was drawn up with the front that B had, and $\frac{y^2}{x}=$ the number of men in rank when B was drawn up with the front that A had,

hence $x+y=84$, (1)

$\frac{x^2}{y}+\frac{y^2}{x}=91$. (2)

Multiplying both members of (2) by xy, we have

$x^3+y^3=91xy$ (3)

Cubing eq. (1) and subtracting (3) from the result, we have $3xy(x+y)=(84)^3-91xy$, but $x+y=84$,

$$252xy=(84)^3-91xy,$$
$$343xy=(84)^3$$
$$\text{or, } 7^3xy=(7\times12)^3=7^3\times12^3$$
$$xy=12^3=1728.$$

Having $x+y=84$, and $xy=1728$, we find $x=48$, and $y=36$, by the method of Form 1, Art. 245.

$\therefore\ x^2=48^2=2304=$ men in column A;
$y^2=36^2=1296=$ " " " B.

FORMULÆ.

Article 252.

NOTE.— Examples 2 to 5 have either been solved before, or are so simple as to require no explanation. We shall, therefore, merely express the respective formula in the form of Rules.

(2) PROBLEM.— To find two numbers, having given the sum of their squares, and the difference of their squares.

RULE.— *Add the difference of the squares to the sum of the squares, multiply the sum by 2 and extract the square root; half the result will be the greater number.*

To find the less number, proceed in the same manner, except that the difference of the squares must be subtracted from their sum.

Ex. The sum of the squares of two numbers is $120\frac{1}{2}$, and the difference of their squares 60; required the numbers.

Ans. $9\frac{1}{2}$, and $5\frac{1}{2}$.

(3) PROBLEM.— Having given the difference of two numbers, and their product, to find the numbers.

RULE.— *To the square of the difference add four times the product, extract the square root of the sum; add the result to the difference, and also subtract the difference from it, then half the sum will be the greater number, and half the difference the less number.*

Ex. The difference of two numbers is 11, and their product 80; required the numbers. *Ans.* 5 and 16.

(4) PROBLEM.— To find a number, having given the sum of the number and its square root.

RULE.— *To the given sum add $\frac{1}{4}$, and extract the square root, subtract the result from the given sum increased by $\frac{1}{2}$, and the remainder will be the required number.*

Ex. The sum of a number and of its square root is $8\frac{3}{4}$; required the number. *Ans.* $6\frac{1}{4}$.

(5) PROBLEM.— To find a number having given the difference of the number and its square root.

RULE.— *To the given difference add $\frac{1}{4}$, extract the square root of the sum, and to the result add the given difference increased by $\frac{1}{2}$; the sum will be the required number.*

Ex. The difference of a number and its square root is $8\frac{3}{4}$; required the number. *Ans.* $12\frac{1}{4}$.

(6) $x+y=s.$ (1)

Squaring, $x^2+2xy+y^2=s^2$,

but $xy=p$, therefore by transposing $2xy$, or $2p$,

$x^2+y^2=s^2-2p$.

Cubing eq. (1) $x^3+3x^2y+3xy^2+y^3=s^3$,

or $x^3+3xy(x+y)+y^3=s^3$,

or $x^3+3ps+y^3=s^3$.

$\therefore\ x^3+y^3=s^3-3ps$.

Again, raising $x+y=s$, to the fourth power,

$x^4+4x^3y+6x^2y^2+4xy^3+y^4=s^4$,

but $4x^3y+6x^2y^2+4xy^3=4xy(x^2+y^2)+6p^2$

$=4p(s^2-2p)+6p^2=4ps^2-2p^2$.

$\therefore\ x^4+y^4=s^4-4ps^2+2p^2$.

As an additional example, let the following problem be proposed:

PROBLEM.—To find two numbers having given their product, and the difference of their cubes.

Let x and y represent the numbers, then

$x^3-y^3=a,$ (1)

$xy=b.$ (2)

Squaring equation (1), adding to the result 4 times the cube of (2) and extracting the square root, we have

$x^3+y^3=a^2+4b^3.$ (3)

Adding together equations (1) and (3), dividing by 2 and extracting the cube root, we find

$x=\sqrt[3]{\left\{\frac{1}{2}(a+\sqrt{a^2+4b^3})\right\}}$.

Similarly, by subtracting equations (1) from (3), we find

$y=\sqrt[3]{\left\{\frac{1}{2}(\sqrt{a^2+4b^3}-a)\right\}}$.

These formulæ give the following

RULE.— *To the square of the difference of the cubes, add four times the cube of their product, extract the square root of the sum; add the result to the difference of the cubes, also subtract the difference from it, then the cube root of one-half the sum will be the greater number, and the cube root of one-half the difference the less number.*

Ex. The difference of the cubes of two numbers is 604, and their product is 45; required the numbers. *Ans.* 5 and 9.

In a similar manner special rules might be formed for the solution of nearly all the questions on page 214 of the Algebra.

SPECIAL SOLUTIONS AND EXAMPLES.

Article 253.

(2) By adding $2x$ to each member, the equation becomes
$x^3-x=2+2x$,
or $x(x^2-1)=2(x+1)$,
divide both members by $x+1$,
$x(x-1)=2$.
Whence $x=-1$ or 2.

(3) Transposing $\frac{4}{9}$ and $\frac{2}{3x}$, the equation becomes

$x^2-\frac{4}{9}=1+\frac{2}{3x}$

or $(x+\frac{2}{3})(x-\frac{2}{3})=\frac{1}{x}(x+\frac{2}{3})$,

$\therefore$ $x+\frac{2}{3}=0$, or $x=-\frac{2}{3}$.

Also, $x-\frac{2}{3}=\frac{1}{x}$.

Whence $x=\frac{1}{3}(1\pm\sqrt{10})$.

(4) Transpose 1 to the left member, then the equatio[illegible] placed under the form
$2x^2(x-1)+x^2-1=0$,
or $2x^2(x-1)+(x+1)(x-1)=0$
$\therefore$ $x-1=0$, or $x=1$.
Also, $2x^2+x+1=0$.
Whence $x=\frac{1}{4}(-1\pm\sqrt{-7})$.

(5) The equation may be placed under the following form
$x^3-2x^2-x^2+2x-x+2=0$,
or $x^2(x-2)-x(x-2)-(x-2)=0$.
$\therefore$ $x-2=0$, or $x=2$.
Also $x^2-x-1=0$.
Whence $x=\frac{1}{2}(1\pm\sqrt{5})$

(6) Multiplying both sides by x, we have
$x^4=6x^2+9x$,

or, $x^4+3x^2=9x^2+9x$,

$x^4+3x^2+\frac{9}{4}=9x^2+9x+\frac{9}{4}=9(x^2+x+\frac{1}{4})$,

$x^2+\frac{3}{2}=\pm 3(x+\frac{1}{2})$,

$x^2=3x$, or $x=3$,

or $x^2+3x=-3$, and $x=\frac{1}{2}(-3\pm\sqrt{-3})$.

Or, thus, $x^3=6x+9$.

$\therefore\ x^3-27=6x-18=6(x-3)$,

dividing by $x-3$, $x^2+3x+9=6$, from which the value of x is readily obtained.

(7) $x+7x^{\frac{1}{3}}-22=(x-8)+7(x^{\frac{1}{3}}-2)=0$,

dividing by $x^{\frac{1}{3}}-2$, we have

$$x^{\frac{2}{3}}+2x^{\frac{1}{3}}+4+7=0$$

$$x^{\frac{2}{3}}+2x^{\frac{1}{3}}=-11$$

$$x^{\frac{2}{3}}+2x^{\frac{1}{3}}+1=-10$$

$$x^{\frac{1}{3}}=-1\pm\sqrt{-10}$$

$$x=(-1\pm\sqrt{-10})^3=29\pm 7\sqrt{-10}.$$

From $x^{\frac{1}{3}}-2=0$, we have $x^{\frac{1}{3}}=2$ and $x=8$

(8) This equation may be written under the form

$x^4-81+\frac{13}{3}(x^2-9)x=0$,

or $(x^2+9)(x^2-9)+\frac{13}{3}(x^2-9)x=0$.

$\therefore\ x^2-9=0$, and $x=+3$, or -3.

Also, $x^2+9+\frac{13}{3}x=0$.

Whence $x=\frac{1}{6}(-13\pm\sqrt{-155})$.

(10) Multiplying both members by x, and adding $x+1$ to each side, we have $x^2-2x+1=4+4\sqrt{x}+x$,

extracting the square root, $x-1=\pm(2+\sqrt{x})$,

From the equation $x-1=2+\sqrt{x}$, by transposing $\sqrt{x}$ and -1, we have

$x-\sqrt{x}=3$.

Whence $\sqrt{x}=\frac{1}{2}\pm\frac{1}{2}\sqrt{13}$, and $x=\frac{1}{2}(7\pm\sqrt{13})$

From the equation $x-1=-2-\sqrt{x}$, similarly we find $\sqrt{x}=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$, and $x=\frac{1}{2}(-1\mp\sqrt{-3})$.

(11) Adding $\frac{1}{x^2}$ to each member, we have

$\frac{49x^2}{4}-49+\frac{49}{x^2}=9+\frac{6}{x}+\frac{1}{x^2}$,

extracting the square root, $\frac{7x}{2}-\frac{7}{x}=\pm\left(3+\frac{1}{x}\right)$.

From the equation $\frac{7x}{2}-\frac{7}{x}=3+\frac{1}{x}$, by clearing of fractions transposing and reducing, we find

$x^2-\frac{6}{7}x=\frac{16}{7}$.

Whence $x=2$, or $-\frac{8}{7}$.

From the equation $\frac{7x}{2}-\frac{7}{x}=-3-\frac{1}{x}$, similarly we find

$x^2+\frac{6}{7}x=\frac{12}{7}$.

Whence $x=\frac{1}{7}(-3\pm\sqrt{93})$.

(12) Transposing $-34x$, and adding $\left(\frac{17x}{4}\right)^2$ to each side, we have $x^4+\frac{17x^3}{2}+\left(\frac{17x}{4}\right)^2=16+34x+\left(\frac{17x}{4}\right)^2$,

extracting the sq. root, $x^2+\frac{17x}{4}=\pm\left(4+\frac{17x}{4}\right)$.

From the equation $x^2+\frac{17x}{4}=4+\frac{17x}{4}$, we have

$x^2=4$, and $x=\pm2$.

From the equation $x^2+\frac{17x}{4}=-4-\frac{17x}{4}$, we have

$$x^2+\frac{17x}{2}=-4.$$

Whence $x=-8$, or $-\frac{1}{2}$.

(13) First $-(3x^2+x)=-3x^2\left(1+\frac{1}{3x}\right)$.

Dividing both members of the equation by x^4, and adding to each side $\frac{9}{4x^4}$, we have

$$\left(1+\frac{1}{3x}\right)^2-\frac{3}{x^2}\left(1+\frac{1}{3x}\right)+\frac{9}{4x^4}=\frac{70}{x^4}+\frac{9}{4x^4}=\frac{289}{4x^4}.$$

extracting the square root, $1+\frac{1}{3x}-\frac{3}{2x^2}=\pm\frac{17}{2x^2}$,

hence $1+\frac{1}{3x}=+\frac{10}{x^2}$, or $-\frac{7}{x^2}$,

clearing of fractions, $x^2+\frac{1}{3}x=+10$, or -7.

Whence $x=3$, or $-3\frac{1}{3}$, or $\frac{1}{6}(-1\pm\sqrt{-251})$.

(14) Multiplying by 2, and adding $\frac{8x}{36}+\frac{81}{36}$ to each side, we have $\frac{36}{x^2}+\frac{18}{x}+\frac{81}{36}=\frac{x^2}{36}+\frac{8x}{36}+\frac{16}{36}$;

extracting the sq. root, $\frac{6}{x}+\frac{9}{6}=\pm\left(\frac{x}{6}+\frac{4}{6}\right)$.

Taking the positive sign, we have the equation

$x^2-5x=36$, from which $x=9$, or -4.

From the equation $\frac{6}{x}+\frac{9}{6}=-\left(\frac{x}{6}+\frac{4}{6}\right)$, we have

the equation $x^2+13x=-36$, from which $x=-9$, or -4.

$\therefore$ the values of x are $+9$, -9, and -4.

(15) Multiplying both sides by 3, transposing $\frac{841}{x^2}$ and $\frac{1}{x^2}$ and adding 1 to each side, we have

$81x^2+18+\frac{1}{x^2}=\frac{841}{x^2}+\frac{232}{x}+16$,

extracting the square root, $9x+\frac{1}{x}=\pm\left(\frac{29}{x}+4\right)$,

Taking the plus sign we find $x=2$, or $-\frac{14}{9}$;

taking the minus sign " " $x=\frac{1}{9}(-2\pm\sqrt{-266})$.

(19) Let $x+y=s$, and $xy=p$,

then $x^2+y^2=s^2-2p$, and $x^3+y^3=s^3-3sp$,

and by substitution the first equation becomes

$2s^3+1=(s^2-2p)(p+s^3-3sp)$,

but $s=x+y=3$, hence by substitution the equation

becomes $55=(9-2p)(p+27-9p)=243-126p+16p^2$,
or $16p^2-126p=-188$.
Whence $p=\frac{47}{8}$ or 2.
Taking $x+y=3$, and $xy=2$, we readily find $x=2$, and $y=1$.

(20 Dividing both sides of the equation by $1+x$, we have
$1-x+x^2=a(1+x)^2=a+2ax+ax^2$,
transposing and reducing,
$(a-1)x^2+(2a+1)x=1-a$

$$\text{or } x^2+\frac{2a+1}{a-1}x=\frac{1-a}{a-1}=-\frac{a-}{a-1}=-1.$$

$$x^2+\frac{2a+1}{a-1}x+\frac{(2a+1)^2}{4(a-1)^2}=\frac{(2a+1)^2}{4(a-1)^2}-1=\frac{12a-3}{4(a-1)^2}.$$

$$x+\frac{2a+1}{2(a-1)}=\frac{\pm\sqrt{12a-3}}{2(a-1)}$$

$$x=\frac{-2a-1\pm\sqrt{12a-3}}{2(a-1)}=\frac{1+2a\pm\sqrt{12a-3}}{2(1-a)}.$$

Since both members of the equation are divisible bv $1+x$, therefore $1+x=0$, and $x=-1$.

(21) $\frac{a}{x^2}-\frac{1}{x}\sqrt{x-2a-\frac{a}{x}}=1$.

Transposing, multiplying by x, and arranging the terms under the radical, we have

$$\frac{a}{x}-x=\sqrt{-\left(\frac{a}{x}-x\right)-2a},$$

$\left(\frac{a}{x}-x\right)^2+\left(\frac{a}{x}-x\right)=-2a$, by squaring and transposing.

Putting $\frac{a}{x}-x=$ a single unknown quantity, and finding its value, we have

$$\frac{a}{x}-x=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{1-8a}=b,$$

whence $a-x^2=bx$, or $x^2+bx=a$,

$$x=-\frac{b}{2}\pm\tfrac{1}{2}\sqrt{b^2+4a}.$$

Substituting the value of b, we find

$b^2=\frac{1}{2}+\frac{1}{2}\sqrt{1-8a}-2a$

$b^2+4a=\frac{1}{4}(2+2\sqrt{1-8a}+8a)$.

$\therefore\ x=\frac{1}{2}(+\frac{1}{2}\pm\frac{1}{2}\sqrt{1-8a})\pm\frac{1}{2}\sqrt{\frac{1}{4}(2+2\sqrt{1-8a}+8a)}$

$$=\frac{1}{4}\left\{1\pm\sqrt{1-8a}\pm\sqrt{2\pm2\sqrt{1-8a}+8a}\right\}.$$

(22) Let $x+y=s$, and $xy=p$, then the equations become

$s+ps+p^2=85$, (1)

$p+s^2+ps=97$, (2)

adding $s^2+2ps+p^2+p+s=182$,

or, $(s+p)^2+(p+s)=182$.

Whence $s+p=+13$, or -14.

Taking $s+p=13$, $s=13-p$, and substituting this in (1)

$13-p+p(13-p)+p^2=85$, whence $p=6$.

$\therefore\ x+y=7$, and $xy=6$, from which we find $x=6$ or and $y=1$ or 6.

(23) $\frac{2c^2}{d^2}+\frac{ac}{d}-(a-b)(2c+ad)\frac{x}{d}=(a+b)\frac{cx}{d}-(a^2-b^2)x^2$.

$2c^2+acd-(a-b)(2c+ad)dx=(a+b)cdx-(a^2-b^2)d^2x^2$,

$(a^2-b^2)d^2x^2-2acdx-a^2d^2x+2bcdx+abd^2x-acdx-bcdx$
$=-acd-2c^2$,

$(a^2-b^2)d^2x^2-3acdx+bcdx-a^2d^2x+abd^2x=-acd-2c^2$

$x^2-\frac{3acd-bcd+a^2d^2-abd^2}{(a^2-b^2)d^2}x=-\frac{acd+2c^2}{(a^2-b^2)d^2}$

$x^2-\frac{(3a-b)cd+(a-b)ad^2}{(a^2-b^2)d^2}x$

$+\frac{(3a-b)^2c^2d^2+(a-b)^2a^2d^4+2acd^3(3a-b)(a-b)}{4(a^2-b^2)^2d^4}$

$=\frac{-4a^3cd^3-8a^2c^2d^2+4ab^2cd^3+8b^2c^2d^2}{4(a^2-b^2)^2d^4}$

$+\frac{(3a-b)^2c^2d^2+(a-b)^2a^2d^4+2acd^3(3a-b)(a-b)}{4(a^2-b^2)^2d^4}$,

$=\frac{(a-3b)^2c^2d^2+(a-b)^2a^2d^4+2acd^3(a-3b)(a-b)}{4(a^2-b^2)^2d^4}$,

$$x-\frac{(3a-b)ca+(a-b)ad^2}{2(a^2-b^2)d^2}=\pm\frac{(a-3b)cd+(a-b)ad^2}{2(a^2-b^2)d^2},$$

$$x=\frac{(3a-b)cd+(a-b)ad^2}{2(a^2-b^2)d^2}\pm\frac{(a-3b)cd+(a-b)ad^2}{2(a^2-b^2)d^2}$$

$$=\frac{4(a-b)cd+2(a-b)ad^2}{2(a^2-b^2)d^2}=\frac{2c+ad}{(a+b)d}$$

$$\text{or }=\frac{2(a+b)cd}{2(a^2-b^2)d^2}=\frac{c}{(a-b)d}.$$

(24) $(x^3+1)(x^2+1)(x+1)=30x^3$,

or, $\left(x^2+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)\left(1+\frac{1}{x}\right)=30$,

or, $\left(x^2+\frac{1}{x^2}+x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)=30$,

Let $x+\frac{1}{x}=s$; $x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2=s^2-2$

$(s^2-2+s)s=30$, or $s^3-2s+s^2=30$,

$s^4+s^3-2s^2=30s$,

$s^4+s^3+\frac{s^2}{4}=\frac{9s^2}{4}+30s$,

$\left(s^2+\frac{s}{2}\right)^2=\frac{9s^2}{4}+30s$

$\left(s^2+\frac{s}{2}\right)^2+10\left(s^2+\frac{s}{2}\right)=\frac{49s^2}{4}+35s$,

$\left(s^2+\frac{s}{2}\right)^2+10\left(s^2+\frac{s}{2}\right)+25=\frac{49s^2}{4}+35s+25$

$s^2+\frac{s}{2}+5=\frac{7s}{2}+5$

$s^2=3s$, and $s=3=x+\frac{1}{x}$,

$x+\frac{1}{x}=3$,

Whence $x=\frac{1}{2}(3\pm\sqrt{5})$.

(25) $x^3+y^3=35$, and $x^2+y^2=13$.

Let $x+y=v$,

$xy=z$,

then $v^3-3vz=35$ (3)

and $v^2-2z=13$, (4)

$2v^3-6vz=70,$ (5) by multiplying (3) by 2

$3v^3-6vz=39v,$ (6) by multiplying (4) by $3v$

$v^3=39v-70,$ by subtracting (5) from (6)

$v^3-39v=-70,$

$v^4-39v^2=-70v,$

$25v^2=25v^2,$

$v^4-14v^2=25v^2-70v,$

$v^4-14v^2+49=25v^2-70v+49,$

$v^2-7=\pm(5v-7),$

$v^2=5v$, and $v=5,$

or $v^2+5v=14$, and $v=+2$, or -7;

but $v^2-2z=13,$

$25-2z=13$, and $z=6,$

or, $4-2z=13$, and $z=-\frac{9}{2},$

or, $49-2z=13$, and $z=18.$

From $\left.\begin{array}{l}x+y=5,\\ xy=6,\end{array}\right\}$ $\begin{array}{l}x=3, \text{ or } 2,\\ y=2, \text{ or } 3.\end{array}$

From $\left.\begin{array}{l}x+y=2,\\ xy=-\frac{9}{2},\end{array}\right\}$ $\begin{array}{l}x=1\pm\frac{1}{2}\sqrt{22}\\ y=1\pm\frac{1}{2}\sqrt{22}.\end{array}$

From $\left.\begin{array}{l}x+y=-7,\\ xy=18,\end{array}\right\}$ $\begin{array}{l}x=-\frac{7}{2}\pm\frac{1}{2}\sqrt{-23},\\ y=-\frac{7}{2}\mp\frac{1}{2}\sqrt{-23},\end{array}$

(26) Let $xyz=p$, and $x+y+z=s$, then the equations become

$$\frac{p}{s-z}=a, \quad (1)$$

$$\frac{p}{s-y}=b, \quad (2)$$

$$\frac{p}{s-x}=c, \quad (3)$$

hence $z=s-\frac{p}{a},$ (4)

$$y=s-\frac{p}{b} \quad (5)$$

$$x=s-\frac{p}{c}, \quad (6)$$

adding, $x+y+z$, or $s=3s-p\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

Whence $s=\frac{(ab+ac+bc)p}{2abc}$.

Substituting this value of s in equations (4), (5), and (6),

we get $z=\frac{(ab+ac-bc)p}{2abc}$, (7)

$$y=\frac{(ab+bc-ac)p}{2abc}, \quad (8)$$

$$x=\frac{(ac+bc-ab)p}{2abc}. \quad (9)$$

Multiplying equations (7), (8), and (9) together, we find

xyz, or $p=\frac{(ac+bc-ab)(ab+bc-ac)(ab+ac-bc)p^3}{8a^3b^3c^3}$,

whence $p=2abc\sqrt{\left\{\frac{2abc}{(ac+bc-ab)(ab+bc-ac)(ab+ac-bc)}\right\}}$,

Substituting this value of p in equations (7), (8), and (9),

we get $x=\sqrt{\left\{\frac{2abc(ac+bc-ab)}{(ab+ac-bc)(ab+bc-ac)}\right\}}$,

$$y=\sqrt{\left\{\frac{2abc(ab+bc-ac)}{(ac+bc-ab)(ab+ac-bc)}\right\}},$$

$$z=\sqrt{\left\{\frac{2abc(ab+ac-bc)}{(ac+bc-ab)(ab+bc-ac)}\right\}}.$$

(27) Dividing both members of (1) by x^3, and both members of (2) by y^3, we have

$$\left(x^3+\frac{1}{x^3}\right)y=y^2+1,$$

or, $x^3+\frac{1}{x^3}=y+\frac{1}{y}$, (3)

and $y^3+\frac{1}{y^3}=9\left(x+\frac{1}{x}\right)$,

or, $\frac{1}{3}\left(y^3+\frac{1}{y^3}\right)=3\left(x+\frac{1}{x}\right)$;

$$\therefore \frac{1}{3}\left(y^3+\frac{1}{y^3}\right)+y+\frac{1}{y}=x^3+3\left(x+\frac{1}{x}\right)+\frac{1}{x^3},$$

$$y^3+\frac{1}{y^3}+3\left(y+\frac{1}{y}\right)=3\left(x+\frac{1}{x}\right)^3,$$

$\therefore\ y+\frac{1}{y}=\left(x+\frac{1}{x}\right)\sqrt[3]{3}$, by extracting the cube root.

And $x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\sqrt[3]{3}$ by (3),

dividing both members by $x+\frac{1}{x}$, we have

$$x^2-1+\frac{1}{x^2}=\sqrt[3]{3},$$

$$x^2+2+\frac{1}{x^2}=\sqrt[3]{3}+3, \text{ by adding } +3,$$

$$x+\frac{1}{x}=\sqrt{\sqrt[3]{3}+3}, \text{ by extracting the square root.}$$

Similarly, $x-\frac{1}{x}=\sqrt{\sqrt[3]{3}-1}$, by subtracting 1.

$$\therefore\ x=\tfrac{1}{2}\left\{\sqrt{\sqrt[3]{3}+3}+\sqrt{\sqrt[3]{3}-1}\right\}.$$

$$\text{But } y+\frac{1}{y}=\left(x+\frac{1}{x}\right)\sqrt[3]{3}=\sqrt[3]{3}\sqrt{\sqrt[3]{3}+3}.$$

$$\text{Whence } y=\tfrac{1}{2}\left\{\sqrt[3]{3}.\sqrt{\sqrt[3]{3}+3}\pm\sqrt{3\sqrt[3]{9}-1}\right\}.$$

RATIO, PROPORTION, AND PROGRESSIONS.

EXERCISES IN RATIO AND PROPORTION

Article 278.

NOTE.— The solutions of these exercises are given, not because they are difficult, but because many of them are of a character not heretofore presented to the notice of Teachers.

(1) 3 to 4 $=\frac{4}{3}$; 3^2 to $4^2=\frac{16}{9}$; $\frac{4}{3}=\frac{12}{9}$, and since $\frac{16}{9}$ is greater than $\frac{12}{9}$, the ratio of 3^2 to 4^2 is greater than the ratio of 3 to 4.

(2) Duplicate ratio of 2 to 3 is 2^2 to $3^2 =$ 4 to 9, triplicate ratio of 3 to 4 is 3^3 to $4^3 =27$ to 64, subduplicate ratio of 64 to 36 is $\sqrt{64}$ to $\sqrt{36} =8$ to 6, $4\times27\times8$ to $9\times64\times6=864$ to $3456=1$ to 4;
Or, by canceling thus,

$$\frac{\not 9\times\not 6\not 4\times 6}{4\times\not 2\not 7\times\not 8}=\frac{\not 8\times\not 6}{\not 4\times\not 3}=\frac{2\times2}{1}=\frac{4}{1}=1 \text{ to } 4.$$

(3) Let $x=$ the quantity, then

$$\frac{n+x}{m+x}=\frac{q}{p},$$

whence $np+px=mq+qx$, and $x=\dfrac{mq-np}{p-q}$.

(4) $\dfrac{b}{a}=2\frac{2}{3}=\frac{8}{3}$; dividing both terms of each fraction by 2

$$\frac{b}{2a}=\frac{4}{3}=1\tfrac{1}{3}.$$

Multiplying both terms of the fractions $\dfrac{b}{a}=\frac{8}{3}$, by $\frac{4}{3}$ we have $\dfrac{4b}{3a}=\frac{8}{3}\times\frac{4}{3}=\frac{32}{9}=3\frac{5}{9}$.

(5) $\dfrac{7b}{a}=5\frac{1}{4}=\frac{21}{4}$; dividing both fractions by 7, $\dfrac{b}{a}=\frac{3}{4}$.

Multiplying both terms of the fractions $\dfrac{b}{a}=\frac{3}{4}$, by $\frac{4}{5}$ we have $\dfrac{b}{a}\times\frac{4}{5}=\frac{3}{4}\times\frac{4}{5}$, or $\dfrac{4b}{5a}=\frac{3}{5}$.

(6) $\dfrac{b}{a}=1\frac{2}{3}=\frac{5}{3}$; $\therefore$ $\dfrac{b}{a+b}=\dfrac{5}{3+5}$, or $\dfrac{b}{a+b}=\frac{5}{8}$;

Since $\dfrac{b}{a}=\frac{5}{3}$; $\therefore$ $\dfrac{a}{b}=\frac{3}{5}$, and $\dfrac{a}{b-a}=\dfrac{3}{5-3}=\frac{3}{2}$

(7) $\dfrac{n}{m}=\frac{4}{7}$, and $\dfrac{m}{n}=\frac{7}{4}$, also $4m=7n$, and $m=\dfrac{7n}{4}$.

$$m-n=\frac{7n}{4}-n=\frac{3n}{4};$$

dividing $6m$ by each member of this equality

$$\frac{6m}{m-n}=6m\div\frac{3n}{4}=\frac{8m}{n}=8\times\frac{7}{4}=14\ ;$$

Also, dividing $5n$ by each member, we have

$$\frac{5n}{m-n}=5n\times\frac{4}{3n}=\frac{20n}{3n}=6\tfrac{2}{3}.$$

(8) $\frac{2m+3n}{m}=2\frac{3}{5}=\frac{13}{5}$;

clearing of fractions $10m+15n=13m$, or $m=5n$,

and $\frac{n}{m}=\frac{1}{5}$, or 5 to 1.

(9) $\frac{n}{m}=\frac{7}{2}$, and $m=\frac{2}{7}n$, also $12m=\frac{24n}{7}$;

dividing $m+n$ by both members of this equality,

$$\frac{m+n}{12m}=(m+n)\div\frac{24n}{7}=\frac{7m+7n}{24n}=\frac{2n+7n}{24n}=\frac{9n}{24n}=\frac{3}{8}.$$

Also, since $n=3\frac{1}{2}m$, $12n=42m$;
dividing $n-2m$ by both members of this equality, and substituting the value of n in the second member, we have

$$\frac{n-2m}{12n}=\frac{n-2m}{42m}=\frac{3\frac{1}{2}m-2m}{42m}=\frac{3m}{84m}=\frac{1}{28}.$$

(10) $\frac{7x-5y}{5y-8x}=6$, or $7x-5y=30y-48x$,

$$55x=35y$$
$$11x=7y,$$
$$\frac{11}{7}=\frac{y}{x},\text{ or } x:y::7:11.$$

(11) $ab=a^2-x^2$, or $a\times b=(a+x)(a-x)$;
whence (Art. 268), $a:a+x::a-x:b$.

(12) $x^2+y^2=2ax$, or $y^2=2ax-x^2$,
or $y\times y=x(2a-x)$;
whence (Art. 268), $x:y::y:2a-x$.

(13) Let $x=$ the number, then
$a+x:b+x::c+x:d+x$;
$\therefore\ (a+x)(d+x)=(b+x)(c+x)$,
or, $ad+ax+dx+x^2=bc+bx+cx+x^2$,

or, $ax-bx-cx+dx=bc-ad$,

or, $(a-b-c+d)x=bc-ad$,

$$x=\frac{bc-ad}{a-b-c+d}.$$

The pupil should verify this answer by using numbers.

(14) Let a, b, c, and d, be four quantities in proportion, and if possible, let x be a number that being added to each will make the resulting four quantities proportionals; then

$a+x : b+x :: c+x : d+x$.

$\therefore (a+x)(d+x)=(b+x)(c+x)$,

or $ad+ax+dx+x^2=bc+bx+cx+x^2$;

whence $x=\frac{bc-ad}{a-b-c+d}$.

But since a, b, c, d, are in proportion (Art. 267), $ad=bc$,

$\therefore x=\frac{bc-bc}{a-b-c+d}=\frac{0}{a-b-c+d}=0$, (Art. 135);

hence there is no number which being added to each will leave the resulting quantities proportional.

(15) Cubing each term of the second proportion, we have

$a^3 : b^3 :: c+x : d+y$,

but $x : y :: a^3 : b^3$.

$\therefore x : y :: c+x : d+y$, by Art. 272.

Placing the product of the means equal to the product of the extremes, and omitting xy on each side, we find

$x=\frac{cy}{d}$.

(16) Let ma and mb be equal multiples of two quantities, a and b; then since $\frac{mb}{ma}=\frac{b}{a}$, we have (Art. 263),

$ma : mb :: a : b$

(17) Let $\frac{a}{n}$ and $\frac{b}{n}$ be like parts of two quantities, a and b; then

$\frac{b}{n}\div\frac{a}{n}=\frac{b}{n}\times\frac{n}{a}=\frac{b}{a}$ is equal to $\frac{b}{a}$, and we have (Art. 263),

$\frac{a}{n} : \frac{b}{n} :: a : b$.

(18) Let $a:b::c:d$, then ma and mc will be equal multiples of the antecedents, and nb and nd equal multiples of the consonants; then it is required to prove that

$ma:mc::a:c$, (1)

and $nb:nd::a:c$. (2)

First, $\frac{mc}{ma}=\frac{c}{a}$ is equal to $\frac{c}{a}$, hence (1) is a true proportion.

Second, $\frac{nd}{nb}=\frac{d}{b}$, but since $a:b::c:d$, we have

$d=\frac{bc}{a}$ (Art. 267), hence $\frac{d}{b}=\frac{bc}{a}\div b=\frac{bc}{a}\times\frac{1}{b}=\frac{c}{a}$,

which is the ratio of a to c, therefore (2) is a true proportion. (Art. 263.)

(19) Since $a:b::c:d$, $\therefore$ $\frac{b}{a}=\frac{d}{c}$ (Art. 263);

but $\frac{mb}{ma}=\frac{b}{a}$, and $\frac{nd}{nc}=\frac{d}{c}$, therefore

$ma:mb::nc:nd$.

Again, if we take the equation $\frac{b}{a}=\frac{d}{c}$, and multiply both sides by $\frac{n}{m}$, we have $\frac{nb}{ma}=\frac{nd}{mc}$, which gives the proportion $ma:nb::mc:nd$, (Art. 263).

(20) Since $a:b::c:d$, $\therefore$ $\frac{b}{a}=\frac{d}{c}$, (Art. 263);

but $\frac{b}{m}\div\frac{a}{m}=\frac{b}{m}\times\frac{m}{a}=\frac{b}{a}$, and $\frac{d}{n}\div\frac{c}{n}=\frac{d}{n}\times\frac{n}{c}=\frac{d}{c}$,

$\therefore$ $\frac{a}{m}:\frac{b}{m}::\frac{c}{n}:\frac{d}{n}$.

Again, if we take the equation $\frac{b}{a}=\frac{d}{c}$, and multiply both members by $\frac{m}{n}$, we have $\frac{mb}{na}=\frac{md}{nc}$;

but $\frac{mb}{na}=\frac{b}{n}\div\frac{a}{m}$, and $\frac{md}{nc}=\frac{d}{n}\div\frac{c}{m}$; that is, the ratio of $\frac{a}{m}$ to $\frac{b}{n}$ is equal to the ratio of $\frac{c}{m}$ to $\frac{d}{n}$, hence

$\frac{a}{m}:\frac{b}{n}::\frac{c}{m}:\frac{d}{n}$.

(21) Let $a : b :: c : d$, (1)

and $e : f :: g : h$, (2)

from (2) by Art. 271, $f : e :: h : g$, (3)

multiplying together the corresponding terms of (1) and (3) (Art. 277), we have

$af : be :: ch : dg$,

whence $\frac{be}{af} = \frac{dg}{ch}$,

but $\frac{be}{af} = \frac{b}{f} \times \frac{e}{a} = \frac{b}{f} \div \frac{a}{e}$, and $\frac{dg}{ch} = \frac{d}{h} \times \frac{g}{c} = \frac{d}{h} \div \frac{c}{g}$;

$\therefore \frac{b}{f} \div \frac{a}{e} = \frac{d}{h} \div \frac{c}{g}$,

whence (Art. 263), $\frac{a}{e} : \frac{b}{f} :: \frac{c}{g} : \frac{d}{h}$.

(22) Let us take the two proportions

$a : b :: c : d$, (1)

and $ma : e :: mc : f$, (2)

in which the antecedents are proportional, since $a : c :: ma : mc$; then it is required to prove that $b : d :: e : f$.

By alternation (Art. 270), proportions (1) and (2) give

$a : c :: b : d$, whence $\frac{c}{a} = \frac{d}{b}$,

and $ma : mc :: e : f$, whence $\frac{mc}{ma}$, or $\frac{c}{a} = \frac{f}{e}$;

$\therefore \frac{d}{b} = \frac{f}{e}$, or $b : d :: e : f$.

(23) Let a and b be the antecedent and consequent of a ratio, and n any given number, then it is required to prove that

$a \pm \frac{a}{n} : b \pm \frac{b}{n} :: a : b$.

$a \pm \frac{a}{n} = \frac{na \pm a}{n} = a\left(\frac{n \pm 1}{n}\right)$, $b \pm \frac{b}{n} = \frac{nb \pm b}{n} = b\left(\frac{n \pm 1}{n}\right)$;

$\therefore$ the ratio of the first term to the second is $b\left(\frac{n \pm 1}{n}\right) \div a\left(\frac{n \pm 1}{n}\right) = \frac{b}{a}$, and since this is the same as the ratio of the third term to the fourth, the proportion is true (Art. 263.)

(24) Developing $(a+b)^2$ and $(a-b)^2$, we have

$a^2+2ab+b^2 : a^2-2ab+b^2 :: b+c : b-c$,

whence (Art. 275), $2a^2+2b^2 : 4ab :: 2b : 2c$,

or (Art. 267), $2c(2a^2+2b^2)=4ab\times 2b=8ab^2$,

or, $4a^2c+4b^2c=8ab^2$,

or, $a^2c+b^2c=2ab^2$,

or, $a^2c=2ab^2-b^2c=b^2(2a-c)$.

$\therefore$ (Art. 268), $a^2 : b^2 :: 2a-c : c$;

by extracting the square root of each term (Art. 276), we have $a : b :: \sqrt{2a-c} : \sqrt{c}$.

Article 279.

(3) By Art. 275, the proportion gives

$2x : 2y :: 4 : 2$, or $4x=8y$, or $x=2y$.

By substituting the value of x in the equation $x^3-y^3=56$ we have $(2y)^3-y^3=56$,

reducing, $7y^3=56$; whence $y=2$, and $x=4$.

(4) From Art. 274, the proportion gives

$x-(x-y) : x :: 6-5 : 6$,

or, $y : x :: 1 : 6$,

whence (Art. 267), $x=6y$.

By substitution the equation becomes $6y\times y^2=384$;

whence $y=4$, and $x=24$.

(5) By Division (Art. 274), the proportion gives

$x+y-x : x :: 7-5 : 5$,

or, $y : x :: 2 : 5$,

whence $x=\frac{5y}{2}$.

By substitution the equation becomes $\frac{5y^2}{2}+y^2=126$,

whence $y=\pm 6$, and $x=\pm 15$.

(6) Extracting the square root of each term of the proportion (Art. 276), we have

$$x+y : x-y :: 8 : 1,$$

(Art. 274), $2x : 2y :: 9 : 7$;

whence $x=\frac{9y}{7}$;

substituting this value of x in the equation, we have

$\frac{9y^2}{7}=63$, whence $y=\pm 7$, and $x=\pm 9$.

(7) Writing b in the form $\frac{b}{1}$, the equation gives the proportion

$a+\sqrt{a^2-x^2} : a-\sqrt{a^2-x^2} : 1 : b$,

(Art. 275), $2a : 2\sqrt{a^2-x^2} :: b+1 : 1-b$,

or, $a : \sqrt{a^2-x^2} :: b+1 : 1-b$,

(Art. 276), $a^2 : a^2-x^2 :: (b+1)^2 : (1-b)^2$.

But, by means of Art. 274, it may easily be shown that in any proportion the first term is to the difference of th first and second, as the third term is to the difference or the third and fourth ; hence,

$$a^2 : a^2-(a^2-x^2) :: (b+1)^2 : (b+1)^2-(b-1)^2$$

or, $a^2 : x^2 :: (b+1)^2 : 4b$,

whence $x^2(b+1)^2=4a^2b$,

$$x^2=\frac{4a^2b}{(b+1)^2},$$

$$x=\pm\frac{2a\sqrt{b}}{b+1}.$$

(8) The equation gives the proportion

$\sqrt{a+x}+\sqrt{a-x} : \sqrt{a+x}-\sqrt{a-x} : b : 1$,

(Art. 275), $2\sqrt{a+x} : 2\sqrt{a-x} :: b+1 : b-1$,

(Art. 276), $a+x : a-x :: (b+1)^2 : (b-1)^2$,

(Art. 275), $2a : 2x :: 2b^2+2 : 4b$,

or, $a : x :: b^2+1 : 2b$,

whence $x=\frac{2ab}{b^2+1}$.

(9) The equation gives the proportion

$a+x+\sqrt{2ax+x^2} \quad a+x : b : 1$,

(Art. 274), $\sqrt{2ax+x^2} : a+x :: b-1 : 1,$

(Art. 271), $a+x : \sqrt{2ax+x^2} :: 1 : b-1,$

(Art. 276), $a^2+2ax+x^2 : 2ax+x^2 :: 1 : b^2-2b+1,$

(Art. 274, Note), $a^2+2ax+x^2 : a^2 :: 1 : 2b-b^2,$

(Art. 276), $a+x : a :: 1 : \pm\sqrt{2b-b^2},$

(Art. 267), $\pm(a+x)\sqrt{2b-b^2}=a,$

$$\pm x\sqrt{2b-b^2}=a\mp a\sqrt{2b-b^2},$$

$$x=\frac{a\mp a\sqrt{2b-b^2}}{\pm\sqrt{2b-b^2}},$$

$$=a\left(\frac{1\mp\sqrt{2b-b^2}}{\pm\sqrt{2b-b^2}}\right).$$

(10) Let $x+y=$ the greater number, and $x-y=$ the less,
then $(x+y)(x-y)=x^2-y^2=320,$
and $(x+y)^3-(x-y)^3=6x^2y+2y^3$; also,
$x+y-(x-y)=2y$, and $(2y)^3=8y^3$.
$\therefore\ 6x^2y+2y^3 : 8y^3 :: 61 : 1,$
or, by dividing the first and second terms by $2y$,
$3x^2+y^2 : 4y^2 : 61 : 1,$
(Art. 267), $3x^2+y^2=244y^2$;
whence $3x^2=243y^2$, and $x=\pm 9y$.
Substituting the value of x^2 in the equation
$x^2-y^2=320$, we have
$81y^2-y^2=320$;
whence $y=\pm 2$, and $x=\pm 18$.
$\therefore\ x+y=\pm 20,$
$x-y=\pm 16.$

Article 280. Ex. 2. Let $x=$ the number, then
$a : x :: a-b : c-x,$
(Art. 267), $ax-bx=ac-ax$

$$2ax-bx=ac,$$

$$x(2a-b)=ac,$$

$$x=\frac{ac}{2a-b}.$$

VARIATION

Article 290

NOTE.—The solutions to these examples are given for the same reason as those following Article 278, not because they are difficult, but because to many Teachers they will be new.

(3) Since y varies as x, let $y=mx$, then since .. $x=2$, $y=4a$, we have $4a=2m$, or $m=2a$,

$\therefore\ y=2ax$

(4) Since y varies as $\frac{1}{x}$, let $y=\frac{m}{x}$, then since if $x=\frac{1}{2}$, $y=8$

we have $8=\frac{m}{\frac{1}{2}}$ or $m=4$.

$\therefore\ y=\frac{4}{x}$.

(5) Let $y^2=m(a^2-x^2)$, then

$\left(\frac{b^2}{a}\right)^2=m[a^2-(a^2-b^2)]$,

$\frac{b^4}{a^2}=mb^2$, or $m=\frac{b^2}{a^2}$;

$\therefore\ y^2=\frac{b^2}{a^2}(a^2-x^2)$, and $y=\frac{b}{a}\sqrt{a^2-x^2}$.

(6) Here we have $y=v+z$, where $v\propto x$, and $z\propto\frac{1}{x^2}$

Let $v=mx$, and $z=\frac{n}{x^2}$, then $y=mx+\frac{n}{x^2}$.

Since $y=6$ when $x=1$, we have $6=m+n$, (1)

and " $y=5$ when $x=2$, " " $5=2m+\frac{n}{4}$. (2)

From equations (1) and (2) we readily find $m=2$ and $n=4$, hence $y=2x+\frac{4}{x^2}$.

(7) Here we have $y=a+v+z$, where a is constant, $v\propto x$, and $z\propto x^2$. Let $v=mx$, and $z=nx^2$

then $y=a+mx+nx^2$.

$\cdot\ 6=a+m+n$, (1

$11=a+2m+4n,$ (2)
$18=a+3m+9n.$ (3)

From the equations (1), (2), (3), by elimination (Art. 158), we find $a=3$, $m=2$ $n=1$ · hence,
$y=3+2x+x^2$.

(8) Since $s\propto t^2$ when is constant; and $s\propto j$ when t is constant, therefore, when both vary is evident from Art 283 (3), that $s\propto ft^2$;
then let $s=mft^2$;
but since $2s=f$, or $s=\frac{1}{2}f$ when $t=1$, therefore
$\frac{1}{2}f=mf1^2$,
whence $m=\frac{1}{2}$,
and $s=\frac{1}{2}ft^2$.

(9) Since $r\propto x$, let $r=nx$,
and since $s\propto\sqrt{x}$, let $s=n\sqrt{x}$.
then $y=mx+n\sqrt{x}$;
hence, by substituting the corresponding values of y and x, we have
$5=4m+2n,$ (1)
$10=9m+3n.$ (2)
From these equations we find $m=\frac{5}{6}$ and $n=\frac{5}{6}$,
hence $y=\frac{5}{6}x+\frac{5}{6}\sqrt{x}=\frac{5}{6}(x+\sqrt{x})$.

(10) Let $x=\frac{p}{y^m}$, and $y=\frac{q}{z^n}$, p and q being invariable,

then $y^m=\frac{q^m}{z^{mn}}$, and $\frac{p}{y^m}=p\div\frac{q^m}{z^{mn}}=\frac{p}{q^m}.z^{mn}$

$\therefore\ x=\frac{p}{q^m}.z^{mn}$.

$a=\frac{p}{q^m}.c^{mn}$, and $\frac{p}{q^m}=\frac{a}{c^{mn}}$.

$\therefore\ x=\frac{a}{c^{mn}}.z^{mn}$,

and $c^{mn}x=az^{mn}$.

It is proper to observe that all the preceding examples adm of proof. Thus, in the answer to example 9, if we substitute for x we ought to find $y=5$, or if we substitute 9 for x we ought to find $y=10$.

The subject of Variation is of considerable use in Natural Philosophy, and though not quite so easily understood as the other parts of Proportion, is worthy the careful study of the learner.

ARITHMETICAL PROGRESSION.

Article 294.

NOTE.— The learner who wishes to understand the subject thoroughly should derive each of the formulæ on page 245, by taking the two equations at the beginning of this article, and finding from them the value of the quantity marked "Requi.ed." We shall illustrate the method of doing this by the solution of two of the most difficult cases, Nos. 2 and 14.

Formula 2. Taking the equations

$$l=a+(n-1)d, \qquad (1)$$

$$\text{and } S=(a+l)\frac{n}{2}, \qquad (2)$$

we have given a, d, and S, and it is required to find l.

The first step is to eliminate n. This may be done by finding the value of n from each of the equations, and putting these values equal to each other.

$$\text{Eq. (1) gives } n=\frac{l-a+d}{d}=\frac{l-a}{d}+1,$$

$$\text{eq. (2) gives } n=\frac{2S}{l+a};$$

$$\therefore \frac{l-a}{d}+1=\frac{2S}{l+a};$$

$$\text{clearing, } l^2-a^2+dl+ad=2dS,$$

$$l^2+dl+\frac{d^2}{4}=2dS+\left(a^2-ad+\frac{d^2}{4}\right)$$

$$=2dS+(a-\tfrac{1}{2}d)^2,$$

$$l+\frac{d}{2}=\pm\sqrt{\{2dS+(a-\tfrac{1}{2}d)^2\}},$$

$$l=-\tfrac{1}{2}d\pm\sqrt{\{2dS+(a-\tfrac{1}{2}d)^2\}}.$$

Formula 14. Here we have the same formulæ, and the same quantities a, a, and S given, to find n.

Finding the value of l in equation (2), and substituting it in (1), we have $a+(n-1)d=\frac{2S-na}{n}$;

clearing and reducing, $n^2+\frac{2a-d}{d}n=\frac{2S}{d}$,

$$n^2+\frac{2a-d}{d}n+\frac{(2a-d)^2}{4d^2}=\frac{(2a-d)^2+8dS}{4d^2};$$

$$\text{whence}\quad n=\pm\frac{\sqrt{(2a-d)^2+8dS}-2a+d}{2d}.$$

Ex. (10) Here $d=-\frac{1}{3}$, and we have given a, d, and n, to find S; we, therefore, use formula 5, whence

$S=\frac{1}{2}n\{2a+(n-1)d\}=\frac{1}{2}n\{26-(n-1)\frac{1}{3}\}$

$=\frac{1}{6}n\{78-(n-1)\} =\frac{1}{6}n(79-n)$.

(11) Here $d=-\frac{7}{6}$, and a, d, and n are given to find S.

$S=\frac{1}{2}n\{1-\frac{7}{6}(n-1)\}=\frac{1}{12}n\{6-7(n-1)\}=\frac{n}{12}(13-7n)$.

(12) Here $a=\frac{2a-b}{a+b}$, and we have a, d, and n given to find S.

Substituting in formula 5, we have

$S=\frac{1}{2}n\left\{\frac{2(a-b)}{a+b}+(n-1)\left(\frac{2a-b}{a+b}\right)\right\}$,

$=\frac{\frac{1}{2}n}{a+b}\{2an-nb-b\}=\frac{n}{a+b}\left\{na-\frac{(n+1)b}{2}\right\}$

(13) Here $d=-\frac{1}{n}$, and we have a, d, and n given, to find S.

$S=\frac{1}{2}n\left\{\frac{2(n-1)}{n}-\frac{1}{n}(n-1)\right\} =\frac{1}{2}n\left\{\frac{n-1}{n}\right\}$

$=\frac{n-1}{2}$.

(14) Here $a=16\frac{1}{12}$, $d=16\frac{1}{12}\times2=32\frac{1}{6}$, and $n=30$, to find l and S.

Formula 1 gives $l=16\frac{1}{12}+(30-1)32\frac{1}{6}=948\frac{11}{12}$.

$S=(l+a)\frac{n}{2}=(948\frac{11}{12}+16\frac{1}{12})\frac{30}{2}=14475$.

(15) Since there are 200 stones, there are 200 terms, therefore $n=200$; and since the person travels $20+20=40$ yards, or 120 feet for the first stone, therefore $a=120$. And since the stones are 2 feet apart, he must travel over *twice* this distance to reach each successive stone, therefore the common difference $d=4$. Applying formula 5 to find the sum of the series of which the first term is 120, the common difference $d=4$, and the number of terms $n=200$, we have

$S=\frac{200}{2}\{2(120)+(200-1)4\}=100(1036)$
$=103600$ feet $=19$m. 4 fur., 640 feet.

(17) Here $a=3$, $b=18$, and $m=4$,

$p=\frac{b-a}{m+1}=\frac{18-3}{4+1}=3$, hence the means are

$3+3=6$, 9, 12, 15.

(18) Here $a=1$, $b=-1$, and $m=9$,

$d=\frac{b-a}{m+1}=\frac{-1-1}{9+1}=-\frac{1}{5}$,

$1-\frac{1}{5}=\frac{4}{5}$, $\frac{4}{5}-\frac{1}{5}=\frac{3}{5}$, &c.

(19) Here $a=19$, $d=-2$, and $S=91$; and it is required to find n, which may be done by formula 14,

where $n=\frac{\pm\sqrt{(2a-d)^2+8dS}-2a+d}{2d}$;

hence $n=\frac{\pm\sqrt{(38+2)^2-16\times 91}-38-2}{-4}$

$=\frac{\pm 12-40}{-4}=\mp 3+10=+13$, or $+7$

Hence, the sum of either 13 terms, or 7 terms, will be equal to 91. To explain the reason of this let the first thirteen terms of the series be written thus,

No. of term,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	13,
terms	19,	17,	15,	13,	11,	9,	7,	5,	3,	1,	—1	—3	-5

Here we see that the sum of the first 7 terms is 91, and the reason that the sum of 13 terms is the same, is owing to the fact that the sum of the last six terms is zero, the sum of the positive and negative quantities being equal to each other.

(20) Here $a=.034$, $d=.0344-.034=.0004$, and $S=2.744$. Substituting these values in formula 14, we have

$$n=\frac{\pm\sqrt{(.068-.0004)^2+.0087936}-.068+.0004}{.008}$$

$$=\frac{\pm.1156-.0676}{.008}=+60.$$

(21) Let $x=$ the first term, and $y=$ the common difference, then $x+y=$ second term, and $x+x+y=2x+y=4$; fifth term $=a+(n-1)d=x+(5-1)y=x+4y=9$.

From these equations we readily find $x=1$ and $y=2$, hence the series is 1, 3, 5, 7, 9, &c.

(22) Let $x=$ the first term, and $y=$ the common difference, then the series is

$x, x+y, x+2y, x+3y, x+4y,$

whence $x+(x+y)=2x+y=18$,

and $(x+2y)+(x+3y)+(x+4y)=3x+9y=12$.

From these equations we find $x=10$, and $y=-2$. It is now required to find n, having given the first term $=10$, the common difference -2, and the sum of the series 28.

$$\text{Formula 14 gives } n=\frac{\pm\sqrt{(20+2)-448}-20-2}{-4}$$

$$=\frac{\mp6-22}{-4}$$

$$=\mp1\tfrac{1}{2}+5\tfrac{1}{2}=4, \text{ or } 7.$$

The series is 10, 8, 6, 4, 2, 0, -2, &c.

Here we readily perceive why the sum of 4 terms is the same as that of 7.

(23) In this example we can readily find the first, second, &c. terms by making $n=1, 2, 3$, &c.

Let $n=1$, then 1^{st} term $=\frac{1}{6}(3-1)=\frac{1}{3}$,

" $n=2$, then 2^{nd} term $=\frac{1}{6}(6-1)=\frac{5}{6}$,

$\frac{5}{6}-\frac{1}{3}=\frac{3}{6}=\frac{1}{2}=$ the common difference.

Sum of n term $=\{\frac{1}{3}+\frac{1}{6}(3n-1)\}\frac{n}{2}=(\frac{1}{2}n+\frac{1}{6})\frac{n}{2}$

$$=\frac{n}{12}(3n+1).$$

(24) Here $a=1$, and $d=2$, to find the sum of r terms, and also of $2r$ terms.

From formula 5, we find the sum of

$$r \text{ terms } =\tfrac{1}{2}r\{2+(r-1)2\}=r^2,$$

$$\text{of } 2r \text{ terms } =\tfrac{1}{2}\times 2r\{2+(2r-1)2\}=4r^2.$$

$$\therefore\ 4r^2 : r^2 :: x : 1,$$

whence $r^2x=4r^2$, and $x=4$.

(25)

The sum S of n terms $=\frac{1}{2}n\{2a+(n-1)d\}=an+\frac{1}{2}n^2d-\frac{1}{2}nd.$

The sum S′ of $2n$ " $=\frac{2n}{2}\{2a+(2n-1)d\}=2an+2n^2d-nd;$

Sum of $2n$ terms — sum of n terms, or the second half of $2n$ terms. $\}$ $=an+\frac{3}{2}n^2d-\frac{1}{2}nd.$

Sum S″ of $3n$ terms $=\frac{3n}{2}\{2a+(3n-1)d\}=3an+\frac{9}{2}n^2d-\frac{3}{2}nd.$

$$\frac{3an+\frac{9}{2}n^2d-\frac{3}{2}nd}{an+\frac{3}{2}n^2d-\frac{1}{2}nd}=3, \text{ the required ratio.}$$

This is an interesting general theorem, which the pupil should illustrate by numbers; thus, if we take the series 1, 3, 5, 7, 9, &c., the sum of the second 4 terms is 48, and the sum of the first 12 terms is 144, being 3 times that of the second half of $2n$ terms n being 4.

(26) $\frac{18}{n+1}=d$, $1+\frac{18}{n+1}=\frac{n+19}{n+1}=$ 1st arithmetical mean.

$a+(n-1)d=\frac{n+19}{n+1}+(n-1)\frac{18}{n+1}=\frac{19n+1}{n+1}=n^{th}$ term.

$a+(n-2-1)d=\frac{n+19}{n+1}+(n-2-1)\frac{18}{n+1}=\frac{19n-35}{n+1}$

$=(n-2)$ term.

$$\left(\frac{19n+1}{n+1}+\frac{n+19}{n+1}\right)\frac{n}{2}=\frac{10n^2+10n}{n+1}=10n=\text{ sum of } n \text{ terms}$$

$$\left(\frac{19n-35}{n+1}+\frac{n+19}{n+1}\right)\frac{(n-2)}{2}=\frac{10n^2-28n+16}{n+1}$$

$=$ sum of $(n-2)$ terms.

$\therefore\ 10n : \frac{10n^2-28n+16}{n+1} :: 5 : 3,$

whence $30n=\frac{50n^2-140n+80}{n+1}$,

or $30n^2+30n=50n^2-140n+80$,

reducing, $2n^2-17n=-8$, whence $n=+8$.

(27) Let $x=$ the number of days the first travels before he is overtaken by the second. It is then required to find the sum of x terms of the arithmetical series whose first term a, is 1, and common difference $d=1$.

$$S=\tfrac{1}{2}n\{2a+(n-1)d\}=\frac{x}{2}\{2+(x-1)\}=\tfrac{1}{2}x^2+\tfrac{1}{2}x.$$

The second travels $(x-5)$ days at the rate of 12 miles a day hence the whole distance he travels is represented by $12(x-5)$.

$\therefore\ \tfrac{1}{2}x^2+\tfrac{1}{2}x=12(x-5)$,
or, $x^2-23x=-120$.
Whence $x=8$ or 15.
and $x-5=3$ or 10.
$\therefore$ the second travels $12\times3=36$ miles,
or $12\times10=120$ miles.

The second traveler overtakes the first at the end of 3 days, when each has traveled 36 miles; the second then passes the first, but as the first increases his speed each day, at the end of the 10^{th} day he overtakes the second and they are thus twice together.

This example furnishes a beautiful illustration of the manner in which the different roots of an equation correspond to the several circumstances of the problem.

GEOMETRICAL PROGRESSION.

Article 300.

NOTE.—All the formulæ in this Article are derived from the two equations

$$l=ar^{n-}, \qquad (1)$$

$$\text{and } S=\frac{ar^n-a}{r-1}=\frac{rl-a}{r-1}, \qquad (2)$$

by supposing any three of the quantities to be known, and then finding the values of the other two. In general, the formulæ are very easily found, but where n is large the resulting numerical equation is hard to solve, and can only be understood by the learner, after he becomes acquainted with tho numerical solution of equations, as contained in the Algebra, Articles 428, to 444. After the pupil becomes acquainted with exponential equations, Articles 382, 383, he will find no difficulty in obtaining the last four formulæ, 17 to 20.

To illustrate the method of finding these formulæ from the two preceding equations, we shall find l, formula 4.

From (1) $a=\frac{l}{r^{n-1}}$,

" (2) $a=rl-S(r-1)$.

Placing these values of a equal to each other, we find

$$l=\frac{S(r^n+r^{n-1})}{r^n-1}=\frac{(r-1)Sr^{n-1}}{r^n-1}.$$

(1) $r=2$, $r^{n-1}=2^7=128$; $ar^{n-1}=5\times128=640$.

(2) $r=\frac{1}{2}$, $r^{n-1}=(\frac{1}{2})^6=\frac{1}{64}$; $ar^{n-1}=54\times\frac{1}{64}=\frac{27}{32}$.

(3) $r=2\frac{1}{4}\div3\frac{3}{8}=\frac{2}{3}$, $r^{n-1}=(\frac{2}{3})^5=\frac{32}{243}$;
$ar^{n-1}=\frac{27}{8}\times\frac{32}{243}=\frac{4}{9}$.

(4) $r=-\frac{14}{21}=-\frac{2}{3}$, $(-\frac{2}{3})^6=\frac{64}{729}$, $\frac{64}{729}\times-21=-\frac{448}{243}$.

(5) $r=\frac{1}{2}\div\frac{1}{3}=\frac{3}{2}$, $r^{n-1}=(\frac{3}{2})^{n-1}=\frac{3^{n-1}}{2^{n-1}}$; $ar^{n-1}=\frac{1}{3}\times\frac{3^{n-1}}{2^{n-1}}=\frac{3^{n-2}}{2^{n-1}}$.

(10) $r=3$, and $l=n^{th}$ term $=1\times3^{n-1}=3^{n-1}$,

$$S=\frac{rl-a}{r-1}=\frac{3\times3^{n-1}-1}{3-1}=\frac{1}{2}(3^n-1).$$

(11) Here $r=-2$, and l, or n^{th} term $=1\times(-2)^{n-1}=\mp 2^{n-1}$ according as n is odd or even.

$$S=\frac{rl-a}{r-1}=\frac{-2\times(\mp 2^{n-1})-1}{-2-1}=\tfrac{1}{3}(1\mp 2^n).$$

(12) Here $r=-\frac{y}{x}$, and l, or n^{th} term $=x\left(-\frac{y}{x}\right)^{n-1}=\frac{x\left(-\frac{y}{x}\right)^n}{-\frac{y}{x}}$

$$=\frac{x^2\left(-\frac{y}{x}\right)^n}{-y};\ S=\frac{rl-a}{r-1}=$$

$$\frac{\frac{-\frac{y}{x}\times x^2\left(-\frac{y}{x}\right)^n}{-y}-x}{-\frac{y}{x}-1}=\frac{x\left(-\frac{y}{x}\right)^n-x}{-\frac{y}{x}-1}$$

$$=\frac{x-x\left(-\frac{y}{x}\right)^n}{\frac{x+y}{x}}=\frac{x^2-x^2\left(-\frac{y}{x}\right)^n}{x+y}=\frac{x^2}{x+y}\left\{1-\left(-\frac{y}{x}\right)^n\right\}$$

(13) Comparing the given quantities with those in formula 13, we have $a=4$, $l=12500$, and $n=6$, to find r.

$$r=\sqrt[n-1]{\frac{l}{a}}=\sqrt[5]{\frac{12500}{4}}=\sqrt[5]{3125}=5.$$

$$S=\frac{rl-a}{r-1}=\frac{12500\times 5-4}{5-1}=15624.$$

(14) Let $x=$ the 1st term, and $y=$ the ratio, then x, xy, and xy^2 represent the first three terms, and

$x+xy=9$, (1)

$x+xy^2=15$. (2)

From these equations we readily find $y=2$, or $-\frac{1}{3}$, hence $x=3$, or $13\frac{1}{2}$; therefore the series is

3, 6, 9, &c.; or, $13\frac{1}{2}$, $-4\frac{1}{2}$, $+1\frac{1}{2}$, &c.

(15) Here $a=\frac{2}{3}$, and $r=\frac{1}{2}$; $S=\frac{a}{1-r}=\frac{\frac{2}{3}}{1-\frac{1}{2}}=\frac{4}{3}$,

(16) Here $a=9$, and $r=\frac{2}{3}$; $S=\frac{9}{1-\frac{2}{3}}=\frac{9}{\frac{1}{3}}=27$.

(17) Here $a=6$ and $r=\frac{1}{3}$; $S=\frac{6}{1-\frac{1}{3}}=\frac{6}{\frac{2}{3}}=9$.

(18) Here $a=\frac{2}{3}$, and $r=-\frac{1}{2}$; $S=\frac{\frac{2}{3}}{1+\frac{1}{2}}=\frac{4}{9}$.

(19) Here $a=100$, and $r=\frac{2}{5}$; $S=\frac{100}{1-\frac{2}{5}}=\frac{100}{\frac{3}{5}}=166\frac{2}{3}$.

(20) Here $a=a$, and $r=\frac{b}{a}$; $S=\frac{a}{1-\frac{b}{a}}=\frac{a^2}{a-b}$

(21) If we begin at the *second* term, the series is a regular geometric series, of which the first term is $2a$, and the ratio $r=a$, hence the sum of this series is $\frac{2a}{1-a}$. Then, adding 1 to this, the sum of the series $1+2a+2a^2+2a^3+$, &c., is $1+\frac{2a}{1-a}=\frac{1-a}{1-a}+\frac{2a}{1-a}=\frac{1+a}{1-a}$.

(22) Let $x=$ the 1st term, and $y=$ the ratio, then

$x+xy=2\frac{2}{3}$, and $S=3=\frac{x}{1-y}$, from the formula $S=\frac{a}{1-r}$.

From these equations we find $y=+\frac{1}{3}$, or $-\frac{1}{3}$, and $x=2$ or 4; hence there are two series, the first being

$$2+\frac{2}{3}+\frac{2}{9}+, \&c.,$$

and the second $4-\frac{4}{3}+\frac{4}{9}-$, &c.

(25) Here $m=2$, and $r=\sqrt[m+1]{\frac{l}{a}}=\sqrt[3]{\frac{2}{\frac{16}{27}}}=\sqrt[3]{\frac{27}{8}}=\frac{3}{2}$.

$\therefore$ $\frac{16}{27}\times\frac{3}{2}=\frac{8}{9}$, and $\frac{8}{9}\times\frac{3}{2}=\frac{4}{3}$, are the means.

(26) Here $m=7$, and $r=\sqrt[8]{\frac{13122}{2}}=\sqrt[4]{\sqrt[2]{6561}}=\sqrt[4]{81}=3$.

$\therefore$ the means are $2\times3=6$, $6\times3=18$, &c.

Article 301.

CIRCULATING DECIMALS

(1) Here $a=\frac{63}{100}=\frac{63}{10^2}$, $r\frac{1}{100}=\frac{1}{10^2}$,

$$S=\frac{.63}{1-\frac{1}{100}}=\frac{.63}{.99}=\frac{63}{99}=\frac{7}{11}.$$

Or, thus, $S= .63636363....$

$100S=63.63636363....$

$99S=63.$

$S=\frac{63}{99}=\frac{7}{11}.$

(2) Here $S=.54123123123....$

$100000S=54123.123123....$

$100S= 54.123123....$

$99900S=54069.$

$S=\frac{54069}{99900}=\frac{18023}{33300}.$

HARMONICAL PROGRESSION.

Article 303.

(3) Inverting the terms 3 and 12, they become $\frac{1}{3}$ and $\frac{1}{12}$.

Let us now insert two arithmetic means between $\frac{1}{3}$ and $\frac{1}{12}$ and the reciprocals of these will be the harmonic means between 3 and 12.

See example 16, page 246. $a=\frac{1}{12}$, $b=\frac{1}{3}$, and $m=2$;

$$\frac{b-a}{m+1}=\frac{\frac{1}{3}-\frac{1}{12}}{2+1}=\frac{3}{12}\div 3=\frac{1}{12},$$

$\frac{1}{12}+\frac{1}{12}=\frac{2}{12}=\frac{1}{6}$, hence 6 is one of the harmonic means ;

$\frac{1}{6}+\frac{1}{12}=\frac{3}{12}=\frac{1}{4}$, hence 4 is the other harmonic mean.

(4) 2 and $\frac{1}{5}$ inverted become $\frac{1}{2}$ and 5. Let us now insert two arithmetic means between $\frac{1}{2}$ and 5.

$$\frac{b-a}{m+1}=\frac{5-\frac{1}{2}}{3}=1\tfrac{1}{2};$$

$\frac{1}{2}+1\frac{1}{2}=2$, hence $\frac{1}{2}$ is one of the harmonic means,

$2+1\frac{1}{2}=3\frac{1}{2}$, hence $\frac{1}{3\frac{1}{2}}=\frac{2}{7}$ is the other

(5) $\frac{1}{2}$ and $\frac{1}{12}$ inverted become 2 and 12, let us now insert 4 arithmetic means between 2 and 12.

$\frac{b-a}{m+1}=\frac{12-2}{4+1}=\frac{10}{5}=2$, hence we have for the arithmetic means, 4, 6, 8, 10, and for the harmonic means,

$$\tfrac{1}{4}, \tfrac{1}{6}, \tfrac{1}{8}, \tfrac{1}{10}.$$

(6) Since a, b, c, are in arithmetical progression, we have $a-b=b-c$; and since b, c, d, are in harmonical progression, we have $\frac{1}{b}$, $\frac{1}{c}$, and $\frac{1}{d}$ in arithmetical progression.

$$\therefore\ \frac{1}{c}-\frac{1}{b}=\frac{1}{d}-\frac{1}{c},$$

or, by reducing the fractions on each side to a common denominator.

$$\frac{b-c}{bc}=\frac{c-d}{cd},$$

multiplying by c, $\frac{b-c}{b}=\frac{c-d}{d}$,

hence (Art. 263), $b : b-c :: d : c-d$,

but $b-c=a-b$, $\therefore\ b : a-b :: d : c-d$,

by Inversion (Art. 271), $a-b : b :: c-d : d$,

by Composition (Art. 273), $a : b :: c : d$, which was required to be proved.

PROBLEMS IN ARITHMETICAL AND GEOMETRICAL PROGRESSION.

Article 304.

(3) Let $x-y$, x and $x+y$, be the numbers,

then $x-y+x+x+y=3x=30$, and $x=10$,

also, $(x-y)^2+x^2+(x-y)^2=3x^2+2y^2=308$.

By substituting the value of x we find $y=2$, hence

$x-y=8$, $x=10$, and $x+y=12$, are the numbers.

(4) Let $x-3y$, $x-y$, $x+y$, and $x+3y$, be the numbers,
then $x-3y+x-y+x+y+x+3y=4x=26$, and $x=6\frac{1}{2}$,
also, $(x-3y)(x+3y)(x-y)(x+y)=880$,
or, $(x^2-9y^2)(x^2-y^2)=880$,
or, $x^4-10x^2y^2+9y^4=880$;

substituting the value of x, and reducing, we find $y=\frac{3}{2}$, hence the numbers are 2, 5, 8, 11.

(5) Let $x=$ the first term, and $y=$ the ratio, then x, xy, xy^2 represent the terms, and

$$x+xy+xy^2=31, \quad (1)$$

$$x+xy : x+xy^2 :: 3 : 13,$$

$$\text{or, } \frac{x+xy^2}{x+xy}=\frac{1+y^2}{1+y}=\frac{13}{3}. \quad (2)$$

From (2) we find $y=5$, and by substituting this in (1), we find $x=1$; therefore, the numbers are 1, 5, and 25.

(6) Let $x-y$, x, and $x+y$, represent the numbers, then

$$(x-y)^2+x^2+(x+y)^2=3x^2+2y^2=83, \quad (1)$$

$$x^2-(x-y)(x+y)=x^2-(x^2-y^2)=y^2=4. \quad (2)$$

From (2) $y=2$, and by substituting this value in (1), we find $x=5$; hence the numbers are 3, 5, 7.

(7) Let $x-3y$, $x-y$, $x+y$, and $x+3y$, represent the numbers then

$$(x-3y)(x+3y)=x^2-9y^2=27, \quad (1)$$

$$(x-y)(x+y)=x^2-y^2=35. \quad (2)$$

From these equations we easily find $y=1$, and $x=6$; hence the numbers are 3, 5, 7, 9.

(8) Let $x-y$, x, and $x+y$, represent the numbers, then
$(x-y)+x+(x+y)=3x=18$, and $x=6$;
also, $2x-2y$, $3x$, and $6x+6y$ are in geometrical progression

$$\therefore\ 2(x-y)(x+y)6=9x^2,$$

$$\text{or, } 12(x^2-y^2)=9x^2,$$

whence $2y=x$, and $y=3$,
therefore the numbers are $6-3=3$, 6, and $6+3=9$.

(9) Let $x-1$, x, and $x+1$, represent the numbers,
then $(x-1)^4+x^4+(x+1)^4=3x^4+12x^2+2=962$;
whence $x=4$, and the numbers are 3, 4, 5.

(10) Let $x-3y$, $x-y$, $x+y$, and $x+3y$, represent the numbers, then $(x-3y)(x-y)(x+y)(x+3y)=(x-3y)(x+3y)(x-y)(x+y)=(x^2-9y^2)(x^2-y^2)=x^4-10x^2y^2+9y^4=840$.

But since the common difference between the numbers is 1, therefore $2y=1$, and $y=\frac{1}{2}$; substituting this value of y and reducing, we find $x=5\frac{1}{2}$; hence the numbers are 4, 5, 6, 7.

(11) Let $x-3y$, $x-y$, $x+y$, and $x+3y$, represent the three numbers then

$$(x-3y)(x-y)(x+y)(x+3y)=x^4-10x^2y^2+9y^4=280, \quad (1)$$

and $(x-3y)^2+(x-y)^2+(x+y)^2+(x+3y)^2=166$,

$$\text{or, } 4x^2+20y^2=166, \quad (2)$$

$\therefore\ x^2=41\frac{1}{2}-5y^2$.

Let $41\frac{1}{2}=a$, then $x^4=a^2-10ay^2+25y^4$.

Substituting the values of x^4 and x^2 in equation (1), and reducing, we have

$$84y^4-830y^2=-\frac{5769}{4}.$$

Whence $y=1\frac{1}{2}$, and by substitution x becomes $5\frac{1}{2}$, whence the numbers are

$5\frac{1}{2}-3(1\frac{1}{2})=1$, $5\frac{1}{2}-1\frac{1}{2}=4$, $5\frac{1}{2}+1\frac{1}{2}=7$, &c.

(12) Let $x-4y$, $x-3y$, $x-2y$, $x-y$, x, $x+y$, $x+2y$, $x+3y$, and $x+4y$, represent the numbers; then their sum $=9x=45$, whence $x=5$;

also, the sum of their squares $=9x^2+60y^2=285$, from which, by substituting the value of x, we find $y=1$; hence the numbers are 1, 2, 3, &c., to 9.

(13) Let $x-3y$, $x-2y$, $x-y$, x, $x+y$, $x+2y$, and $x+3y$, represent the numbers; then their sum $=7x=35$, whence $x=5$ also, the sum of their cubes $=7x^3+84xy^2=1295$, from which, by substituting the value of x, we find $y=1$, hence the numbers are 2, 3, &c., to 8.

(14) Let x and y represent the numbers, then

$$\frac{x+y}{2} : \sqrt{xy} :: 5 : 4,$$

or, $x+y : 2\sqrt{xy} :: 5 : 4$,

(Art. 276), $x^2+2xy+y^2 : 4xy :: 25 . 16$, (Art. 274, Note,) $x^2-2xy+y^2 : x^2+2xy+y^2 :: 9 : 25$,

(Art. 276), $x-y : x+y :: 3 : 5$,

(Art. 275), $2x : 2y :: 8 : 2$,

or, $x : y :: 4 : 1$.

This theorem may also be proved by multiplying together the means and extremes of the first proportion and finding the value of x in terms of y, by which we find $x=4y$ or $\frac{1}{4}y$.

The converse of the preceding proposition is also true ; that is, if one of two numbers is 4 times the other, then their arithmetic mean is to their geometric mean as 5 to 4. Thus, let a and $4a$ be two numbers, then $2\frac{1}{2}a$ is their arithmetic mean, and $2a$ their geometric mean, and $2\frac{1}{2}a : 2a :: 5 : 4$.

(15) Let x^2, xy, and y^2, represent the numbers, then

$$x^2+xy+y^2=7, \qquad (1)$$

$$\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{y^2}=\frac{7}{4}. \qquad (2)$$

Multiplying both members of equation (2) by x^2y^2, we have $x^2+xy+y^2=\frac{7}{4}x^2y^2$, (3)

$\therefore\ \frac{7}{4}x^2y^2=7$, whence $x^2y^2=4$, and $xy=2$.

Substituting the value of xy in (1), we find $x^2+y^2=5$ then from this, and $xy=2$, we readily find $x=2$ and $y=1$; hence the numbers are 4, 2, and 1.

(16) Let $\frac{x^2}{y}$, x, y, and $\frac{y^2}{x}$, represent the numbers,

then $\frac{x^2}{y}+y=10, \qquad (1)$

$x+\frac{y^2}{x}=30. \qquad (2)$

Clearing these equations of fractions, by multiplying (1) by y, and (2) by x, we have

$x^2+y^2=10y$, and

$x^2+y^2=30x$.

whence $10y=30x$, and $y=3x$.

Substituting this value of y in either of the equations (1) and (2), we find $x=3$; hence $y=9$, and the numbers are 1, 3, 9, 27.

(17) Let x, xy, xy^2, xy^3, be the numbers; then $x+xy^3=35$, and $xy+xy^2=30$.

Dividing one equation by the other;

$$\frac{x+xy^3}{xy+xy^2}=\frac{35}{30}, \text{ or } \frac{1+y^3}{y+y^2}=\frac{7}{6}.$$

But $1+y^3$ is divisible by $1+y$, and $y+y^2=y(1+y)$

$$\therefore \frac{1+y^3}{y+y^2}=\frac{(1+y)(1-y+y^2)}{y(1+y)}=\frac{1-y+y^2}{y}=\frac{7}{6};$$

whence $6y^2-13y=-6$, and $y=\frac{3}{2}$ or $\frac{2}{3}$.

And $x=\dfrac{30}{y+y^2}=8$ or 27.

Hence the numbers are 8, 12, 18, 27.

(18) Let x, xy, xy^2, xy^3, be the numbers when increased;

$\therefore$ $x-2$, $xy-4$, xy^2-8, xy^3-15 are in arithmetical progression; hence $1^{st}+3^{rd}=2^{nd}\times 2$; and $2^{nd}+4^{th}=3^{rd}\times 2$;

$\therefore$ $(x-2)+(xy^2-8)=2(xy-4)$;

or, $x-2xy+xy^2=2$; $\therefore$ $x(1-2y+y^2)=2$, (1)

also, $(xy-4)+(xy^3-15)=2(xy^2-8)$;

or, $xy-2xy^2+xy^3=3$; $\therefore$ $xy(1-2y+y^2)=3$. (2)

Dividing equation (2) by (1), we have

$$\frac{xy(1-2y+y^2)}{x(1-2y+y^2)}=\frac{3}{2}, \text{ or } y=\frac{3}{2},$$

whence $x(1-3+\frac{9}{4})=2$.

$\therefore$ $x=8$, $xy=12$, $xy^2=18$, and $xy^3=27$;

and subtracting 2, 4, 8, and 15 from these numbers, the remainders 6, 8, 10, 12, are the numbers required.

(19) Let x, xy, xy^2, be the numbers,

then $x\times xy\times xy^2=x^3y^3=64$,

$$xy=\sqrt[3]{64}=4;$$

also, $x^3+x^3y^3+x^3y^6=584$,

$$x^3+x^3y^6=584-x^3y^3=520.$$

From the equation $xy=4$, we have $x=\dfrac{4}{y}$;

substituting this value of x in the last equation, we have

$\frac{64}{y^3}+64y^3=520.$

dividing by 8, $\frac{8}{y^3}+8y^3=65$;

clearing, $8y^6-65y^3=-8$;

whence (Art. 242), $y^3=8$ or $\frac{1}{8}$, and $y=2$ or $\frac{1}{2}$

$\therefore$ $x=2$ or 8, and the numbers are 2, 4, 8.

PERMUTATIONS, COMBINATIONS, AND BINOMIAL THEOREM.

Articles 305—309.

(1) (Art. 306), $P_2=n(n-1)=5(5-1)=20$;

$P_3=n(n-1)(n-2)=5\times4\times3=60$;

$P_4=n(n-1)(n-2)(n-3)=5\times4\times3\times2=120$

(2) (Art. 308), $C_2=\frac{n(n-1)}{1\times2}=\frac{5\times4}{1\times2}=10$;

$C_3=\frac{n(n-1)(n-2)}{1\times2\times3}=\frac{5\times4\times3}{1\times2\times3}=10$;

$C_4=\frac{n(n-1)(n-2)(n-3)}{1\times2\times3\times4}=\frac{5\times4\times3\times2}{1\times2\times3\times4}=5$;

$C_5=\frac{n(n-1)(n-2)(n-3)(n-4)}{1\times2\times3\times4\times5}$

$=\frac{5\times4\times3\times2\times1}{1\times2\times3\times4\times5}=1.$

(3) (Art. 306a), $P_r=P_3=1\times2\times3=6$.
Thus, NOT, NTO, ONT, OTN, TNO, TON.
$P_5=1\times2\times3\times4=24$.

(4) This is a case of permutations, when all the letters are taken together (Art. 306a).
$\therefore$ $P_6=1\times2\times3\times4\times5\times6=720$.

(5) This is similar to the preceding.
$\therefore$ $P_7=1\times2\times3\times4\times5\times6\times7=5040$.

(6) The whole number of arrangements is evidently equal to the sum of the different permutations of six letters taken 1 together, 2 together, and so on

$P_1=n=$ 6
$P_2=n(n-1)=6\times5=$ 30
$P_3=n(n-1)(n-2)=6\times5\times4=$ 120
$P_4=n(n-1)(n-2)(n-3)=6\times5\times4\times3=$ 360
$P_5=n(n-1)(n-2)(n-3)(n-4)=6\times5\times4\times3\times2=$ . . 720
$P_6=n(n-1)(n-2)(n-3)(n-4)(n-5)=6\times5\times4\times3\times2$
$\times1=$ 720

Ans. . . . . 1956.

(7) Here the number of different products will evidently be equal to the number of combinations of 4 things taken 2 together.

$$\therefore\ C_2=\frac{n(n-1)}{1\times2}=\frac{4\times3}{1\times2}=6.$$

Let the learner verify this result by finding the different products ; they are 12, 15, 18, 20, 24, 30.

(8) Here it is merely required to find the number of combinations of 5 things, taken 3 together.

$$\therefore\ C_3=\frac{n(n-1)(n-2)}{1\times2\times3}=\frac{5\times4\times3}{1\times2\times3}=10.$$

(9) The number of permutations of n things, taken 4 together is $P_4=n(n-1)(n-2)(n-3)$;
taken 3 together, is $P_3=n(n-1)(n-2)$;
$\therefore\ n(n-1)(n-2)(n-3)=6n(n-1)(n-2)$;
dividing each member by $n(n-1)(n-2)$, we have
$n-3=6$, or $n=9$.

(10) By Art. 306, the number of permutations of 15 things taken r together, and $r-1$ together, is

$P_r\ =15\times14\times13\times12$. . . . $(15-\overline{r-2})(15-\overline{r-1})$,
$P_{r-1}=15\times14\times13\times12$. . . . . $(15-\overline{r-2})$.

Here we see that the two quantities are the same, except the last factor of the first quantity, which, by the terms of the question must therefore be equal to 10 ; that is.

$$10=15-\overline{r-1},$$

whence $r=6$.

Thus the permutations of 15 letters, taken 6 together, are $15\times14\times13\times12\times11\times10$, and the permutations of 15 letters, taken 5 together, are $15\times14\times13\times12\times11$, whence it is readily seen that the former is equal to 10 times the latter.

(11) $C_1=n=$ 4

$C_2=\frac{n(n-1)}{1\times1}=\frac{4\times3}{1\times2}=$ 6

$C_3=\frac{n(n-1)(n-2)}{1\times2\times3}=\frac{4\times3\times2}{1\times2\times3}$ 4

$C_4=\frac{n(n-1)(n-2)(n-3)}{1\times2\times3}=\frac{4\times3\times2\times1}{1\times2\times3\times4}=$ 1

Ans. 15.

The learner may easily verify this result by taking the coins, or by finding the different sums that can be formed of the numbers 1, 3, 5, 10 ; the sums are

1, 3, 5, 10 ; 4, 6, 11, 8, 13, 15 ; 9, 14, 16, 18 ; 19

(12) Here it will be necessary to find the different combinations of six things taken singly, two together, three together, four together, five together, and six together.

$C_1=n=$ 6

$C_2=\frac{n\times(n-1)}{1\times2}=\frac{6\times5}{1\times2}=$ 15

$C_3=\frac{n(n-1)(n-2)}{1\times2\times3}=\frac{6\times5\times4}{1\times2\times3}=$ 20

$C_4=\frac{n(n-1)(n-2)(n-3)}{1\times2\times3\times4}=\frac{6\times5\times4\times3}{1\times2\times3\times4}=$ 15

$C_5=\frac{n(n-1)(n-2)(n-3)(n-4)}{1\times2\times3\times4\times5}=\frac{6\times5\times4\times3\times2}{1\times2\times3\times4\times5}=$ . . . 6

$C_6=\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\times2\times3\times4\times5\times6}=\frac{6\times5\times4\times3\times2\times1}{1\times2\times3\times4\times5\times6}=$ 1

Ans. 63.

In this solution we notice an illustration of the principle of Art. 309. Thus the number of combinations of 6 things, taken 1 together, is the same as when taken (6—1), or 5 together ; the number, when taken 2 together, is the same as when taken (6—2), or 4 together.

(13) He may vote for 1 candidate only, or for any 2, or for any 3; hence the whole number of ways in which he can vote will be equal to the number of combinations of four things taken *singly*, of four things taken *two* together and of four things taken *three* together; thus,

$$C_1=n= \dots\dots\dots\dots \quad 4$$

$$C_2=\frac{n(n-1)}{1\times2}=\frac{4\times3}{1\times2}= \dots\dots\dots \quad 6$$

$$C_3=\frac{n(n-1)(n-2)}{1\times2\times3}=\frac{4\times3\times2}{1\times2\times3}= \dots\dots \quad 4$$

Total number of ways $= \dots\dots$ 14.

(14) If we reserve *a*, and take the different combinations of the four remaining letters *b*, *c*, *d*, *e*, taken two together, we may then unite *a* to each of them, hence the required number will be obtained by finding the different combinations of *four* letters taken *two* together.

$C_2=\frac{n(n-1)}{1\times2}=\frac{4\times3}{1\times2}=6$; and the combinations are

abc, abd, abe, acd, ace, ade.

(15) A different guard may be posted as often as there are different combinations of 4 men out of 16.

$$C_4=\frac{n(n-1)(n-2)(n-3)}{1\times2\times3\times4}=\frac{16\times15\times14\times13}{1\times2\times3\times4}=1820.$$

To find the number of times any particular man will be on guard, it is merely necessary to find the different combinations of (4—1)=3 men that can be formed out of (16—1)=15 men, since the reserved man may be combined with each combination of 3 men, giving a combination of 4 men.

$$C_3=\frac{n(n-1)(n-2)}{1\times2\times3}=\frac{15\times14\times13}{1\times2\times3}=455.$$

(16) $C_4=\frac{n(n-1)(n-2)(n-3)}{1\times2\times3\times4}$,

$C_2=\frac{n(n-1)}{1\times2}$,

$$\frac{n(n-1)(n-2)(n-3)}{1\times2\times3\times4} \cdot \frac{n(n-1)}{1\times2} : \ 15:2.$$

(Art. 267), $\frac{2n(n-1)(n-2)(n-3)}{1\times2\times3\times4}=\frac{15n(n-1)}{1\times2}$,

Dividing both members by $\frac{n(n-1)}{1\times2}$,

$$\frac{2(n-2)(n-3)}{3\times4}=15,$$

reducing, $n^2-5n=84$, and $n=12$.

(17) To find the number of peals that may be rung with 5 bells out of 8, find the number of different combinations of 5 things out of 8, then each combination will give as many changes as there are permutations of 5 bells, and the whole number of changes will be equal to the number of combinations multiplied by the number of permutations in each combination.

$$C_5=\frac{n(n-1)(n-2)(n-3)(n-4)}{1\times2\times3\times4\times5}=\frac{8\times7\times6\times5\times4}{1\times2\times3\times4\times5}=56\ ;$$

$P_5=1\times2\times3\times4\times5=120$;

$56\times120=6720$.

The number of changes with the whole peal will evidently be equal to the number of permutations of 8 things taken all together.

$P_8=1\times2\times3\times4\times5\times6\times7\times8=40320$.

(18) Had the letters been different the number would be $P_7=1\times2\times3\times4\times5\times6\times7=5040$; but there are 2 a's, and therefore (Art. 307), we must divide by 1×2 ; $5040\div2=2520$. *Ans.*

(19) Since there are 3 a's, 4 b's and 2 c's, in all 9 letters :

$\therefore$ (Art. 307a), the number of ways is

$$\frac{1\times2\times3\times4\times5\times6\times7\times8\times9}{(1\times2\times3)(1\times2\times3\times4)(1\times2)}=5\times7\times4\times9=1260.$$

(20) The number of terms in which a^3 will stand first, will evidently be equal to the number of permutations take

all together of the letters in b^4c^2, which, by Art. 307 since there are 6 letters, is

$$\frac{1\times2\times3\times4\times5\times6}{(1\times2\times3\times4)(1\times2)}=15.$$

(21) Reserving two letters, there are 5 letters remaining, then each permutation of 5 letters may be preceded by *ab*, therefore the whole number of permutations of 7 letters in which *ab*, or any other combination stands first, will be equal to the whole number of permutations of the remaining letters taken all together.

$1\times2\times3\times4\times5=120.$

When *abc* stands first there are 4 letters remaining;

$1\times2\times3\times4=24.$

When *abcd* stands first there are 3 letters remaining;

$1\times2\times3=6.$

Thus, *abcd(efg)*, *abcd(egf)*, *abcd(feg)*, *abcd(fge)*, *abcd(gef)*, *abcd(gfe)*.

(22) The number of different combinations of 2 consonants, out of 17, is $\frac{n(n-1)}{1\times2}=\frac{17\times16}{1\times2}=136.$

Each of these combinations may be united with each of the 5 vowels, giving $136\times5=680$ different combinations of 2 consonants and 1 vowel; now each of these combinations of 3 letters will give $1\times2\times3=6$ permutations, therefore the whole number of words will be $680\times6=4080$.

(23) In the word " Notation " there are 5 *different* letters; and the number of different combinations of 5 letters, taken 3 together, is $\frac{n(n-1)}{1\times2}=\frac{5\times4}{1\times2}=10$. But there are 2 *n*'s, 2 *o*'s, and 2 *t*'s, each of which pairs may be combined with each of the other 4 letters, and form 4 combinations of 3 letters, making altogether 3×4, or 12 such combinations where the letters are repeated.

$\therefore$ the number required $=10+12=22$.

The learner may easily write out the several combinations; thus the first ten formed of the letters " NOTAI " may be arranged as in Art. 309, and the remaining twelve are *nno, nnt, nna, nni*; *oon, oot, ooa, ooi*; *ttn, tto, tta tti*.

REMARK.— The term "different" is sometimes used in the preceding solutions in connection with combinations; this is not intended, however, to change the meaning of the word combinations, as given in the Algebra (Art. 308), but merely to render it more emphatic.

BINOMIAL THEOREM,

WHEN THE EXPONENT IS A POSITIVE INTEGER

(2) By comparing the quantities with those in the formula (Art. 310, Cor. 3), we find $n=10$, $n-r+1=6$, $a=x$ and $x=y$.

Since $n-r+1=6$, we have $10-r+1=6$, and $r=5$; hence $n-r+2=10-5+2=7$, and $r-1=4$; therefore the coëfficient of the r^{th} term, that is, the term in which the exponent of the leading letter is 6, is

$$\frac{n(n-1)(n-2)(n-3)}{1\times2\times3\times4}=\frac{10\times9\times8\times7}{1\times2\times3\times4}=210. \quad \textit{Ans.}$$

The coëfficient, however, is most readily found by writing out the whole development, thus,

$$(x+y)^{10}=x^{10}+10x^9y+45x^8y^2+120x^7y^3+210x^6y^4+, \text{\&c.}$$

(3) If instead of a, x, n, and r, we substitute c^2, $-d^2$, 12, and 5 in the formula, Cor 3, Art. 310, we have

$$\frac{12\times11\times10\times9}{1\times2\times3\times4}(c^2)^8(-d^2)^4=495c^{16}d^8.$$

(4) Comparing the quantities with those in the formula, Cor. 3, Art. 310, we have $a=a^3$, $x=3ab$, $n=9$, and $r=7$.

$\therefore$ the 7^{th} term is $\dfrac{9\times8\times7\times6\times5\times4}{1\times2\times3\times4\times5\times6}(a^3)^3(3ab)^6$

$=84a^9\times729a^6b^6=61236a^{15}b^6$.

(5) Referring to the same formula, $a=3a^2$, $x=-7x^3$, $n=8$ and $r=5$; $\therefore$ the 5^{th} term $=\dfrac{8\times7\times6\times5}{1\times2\times3\times4}(3a^2)^4(-7x^3)^4$

$=70\times81a^8\times2401x^{12}=13613670a^8x^{12}$.

(6) Here $a=ax$, $x=by$, $n=10$, and $r=6$;

$\therefore$ the 6^{th} term $=\dfrac{10\times9\times8\times7\times6}{1\times2\times3\times4\times5}(ax)^5(by)^5$

$=252a^5b^5x^5y^5$.

(7) Since the exponent of the binomial is 12, there will be 13 terms (Art. 310, Cor. 4), hence the middle term will be the 7^{th}, and $a=a^m$, $x=x^n$, $n=12$, and $r=7$, (Art. 310, Cor. 3);

$$\therefore \text{ the middle term } =\frac{12\times11\times10\times9\times8\times7}{1\times2\times3\times4\times5\times6}(a^m)^6(x^n)^6$$

$$=924a^{6m}x^{6n}.$$

(8) Since the exponent of the binomial is 13, there will be 14 terms, and the two middle terms will be the 7^{th} and 8^{th}, the coëfficients of which will be the same, (Art. 310, Cor. 5).

(Art. 310, Cor. 3), $a=a$, and $x=x$, $n=13$, and $r=7$;

$$\therefore\ 7^{th} \text{ term } =\frac{13\times12\times11\times10\times9\times8}{1\times2\times3\times4\times5\times6}a^7x^6=1716a^7x^6;$$

Since the exponent of the leading letter increases by unity in each term, and the exponent of the other letter decreases by unity, $\therefore$ 8^{th} term $=1716a^6x^7$.

(9) (Art. 310, Cor. 3), $a=1$, $x=x$, $n=11$, $r=8$;

$$\therefore\ 8^{th} \text{ term } =\frac{11\times10\times9\times8\times7\times6\times5}{1\times2\times3\times4\times5\times6\times7}(1)^4(x)^7=330x^7.$$

(10) (Art. 310, Cor. 3), $a=x$, $x=-y$, $n=30$, $r=6$;

$$\therefore\ 6^{th} \text{ term } =\frac{30\times29\times28\times27\times26}{1\times2\times3\times4\times5}(x)^{25}(-y)^5$$

$$=-142506x^{25}y^5.$$

(11) Comparing this with the general expansion of $a+x$ Art. 310, we have $a=3ac$, $x=-2bd$, and $n=5$; and we have

$$(3ac-2bd)^5=(3ac)^5+5(3ac)^4(-2bd)$$
$$+10(3ac)^3(-2bd)^2+10(3ac)^2(-2bd)^3+5(3ac)(-2bd)^4$$
$$+(-2bd)^5=243a^5c^5-810a^4c^4bd+1080a^3c^3b^2d^2$$
$$-720a^2c^2b^3d^3+240acb^4d^4-32b^5d^5.$$

(12) $(a+2b-c)^3=\{(a+2b)-c\}^3=(a+2b)^3-3(a+2b)^2c$

$$+3(a+2b)c^2-c^3=a^3+6a^2b+12ab^2+8b^3$$
$$-3a^2c-12abc-12b^2c+3ac^2+6bc^2-c^3.$$

(13) Since the coëfficients in the expansion of $(a+x)^n$ do not contain either a or x, they will be the same when $a=1$

or $x=1$, or both a and x at the same time $=1$. (See Art 310, Cor. 6).

For the sake of brevity let the coëfficients of the expansion of $(1+x)^n$ be represented by A_1, A_2, A_3, &c. then

$(1+x)^n=1+A_1x+A_2x^2+A_3x^3+A_4x^4+A_5x^5+$, &c.

Writing $-x$, instead of x,

$(1-x)^n=1-A_1x+A_2x^2-A_3x^3+A_4x^4-A_5x^5+$, &c.

Now if x be made $=1$, then since $(1-1)^n=0$, we have

$1-A_1+A_2-A_3+A_4-A_5+$, &c., $=0$.

$\therefore\ 1+A_2+A_4+A_6+$, &c., $=A_1+A_3+A_5+$, &c.

That is, the sum of the coëfficients of the odd terms is equal to the sum of the coëfficients of the even terms.

INDETERMINATE COEFFICIENTS; BINOMIAL THEOREM WHEN THE EXPONENT IS FRACTIONAL OR NEGATIVE; SERIES.

INDETERMINATE COEFFICIENTS

Articles 314—318.

(1) Let $\frac{1+2x}{1-3x}=A+Bx+Cx^2+Dx^3+Ex^4+$, &c.

Clearing of fractions, we have

$1+2x=A+(B-3A)x+(C-3B)x^2+(D-3C)x^2+$, &c.,

from which, by equating the coëfficients of the same powers of x,

$A=1$;

$B-3A=2$, whence $B=5$;

$C-3B=0$, whence $C=15$;

$D-3C=0$, whence $D=45$, &c.

$\therefore\ \frac{1+2x}{1-3x}=1+5x+15x^2+45x^3+$, &c.

(2) Let $\frac{1+2x}{1-x-x^2}=A+Bx+Cx^2+Dx^3+Ex^4+$, &c

Clearing, $1+2x=A+(B-A)x+(C-A-B)x^2$
$+(D-B-C)x^3+$, &c.

$\therefore$ (Art. 314), $A=1,$

$B-A=2$, whence $B=3$;

$C-A-B=0$, whence $C=4$;

$D-B-C=0$, whence $D=7$;

$$\therefore \frac{1+2x}{1-x-x^2}=1+3x+4x^2+7x^3+11x^4+, \&c.$$

Here we easily perceive that the law is, that the coëfficient of any term is equal to the sum of the coëfficients of the two preceding terms.

(3) Let $\frac{1-3x+2x^2}{1+x+x^2}=A+Bx+Cx^2+Dx^3+$, &c.

Clearing, $1-3x+2x^2=A+(A+B)x+(A+B+C)x^2+(B+C+D)x^3+$, &c.

$\therefore$ (Art. 314), $A=\ 1$;

$A+B=-3$, whence $B=-3-A=-4$;

$A+B+C=\ 2$, whence $C=2-B-A=5$;

$B+C+D=\ 0$, whence $D=-B-C=-1$;

$C+D+E=\ 0$, whence $E=-C-D=-4$; &c.

$\therefore$ the series is $1-4x+5x^2-x^3-4x^4+$, &c.

(4) Let $\frac{3+2x}{5+7x}=A+Bx+Cx^2+Dx^3+Ex^4+$, &c.

Clearing, $3+2x=5A+(7A+5B)x+(7B+5C)x^2+(7C+5D)x^3+$, &c.

$\therefore$ (Art 314), $5A=3$, whence $A=\frac{3}{5}$;

$7A+5B=2$, whence $B=-\frac{11}{25}=-\frac{11}{5^2}$;

$7B+5C=0$, whence $C=\frac{77}{5^3}=\frac{7.11}{5^3}$;

$7C+5D=0$, whence $D=-\frac{7^2.11}{5^4}$; &c.

$\therefore$ the series is $\frac{3}{5}-\frac{11}{5^2}x+\frac{7.11}{5^3}x^2-\frac{7^2.11}{5^4}x^3+$, &c.

(5) Let $\frac{1+x}{(1-x)^3}=A+Bx+Cx^2+Dx^3+Ex^4+$, &c

Clearing, by multiplying both sides by $(1-x)^3$,

$$1+x=A+(B-3A)x+(3A-3B+C)x^2$$
$$+(3B-A-3C+D)x^3+(3C-B-3D+E)x^4+, \&c.$$

$\therefore$ (Art. 314), $A=1$;

$B-3A=1$, whence $B=4=2^2$;

$3A-3B+C=0$, whence $C=9=3^2$;

$3B-A-3C+D=0$, whence $D=16=4^2$,

$3C-B-3D+E=0$, whence $E=25=5^2$; &c.

$\therefore$ the series is $1^2+2^2x+3^2x^2+4^2x^3+5^2x^4+$, &c.

(6) Let $\sqrt{1-x}=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+$, &c.
Squaring both members,

$$1-x=A^2+2ABx+(2AC+B^2)x^2+(2AD+2BC)x^3$$
$$+(2AE+2BD+C^2)x^4+, \&c.$$

$\therefore$ (Art. 314), $A^2=\ \ 1$, whence $A=1$;

$2AB=-1$, whence $B=-\frac{1}{2}$;

$2AC+B^2=\ \ 0$, whence $C=-\frac{1}{8}=-\frac{1}{2\cdot 4}$;

$2AD+2BC=\ \ 0$, whence $D=-\frac{1}{16}=-\frac{3}{2\cdot 4\cdot 6}$;

$2AE+2BD+C^2=\ \ 0$, whence $E=-\frac{5}{128}=-\frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 8}$,

$\therefore$ the series is $1-\frac{x}{2}-\frac{x^2}{2\cdot 4}-\frac{3x^3}{2\cdot 4\cdot 6}-\frac{3\cdot 5x^4}{2\cdot 4\cdot 6\cdot 8}-$, &c.

(7) If we assume $(1+x+x^2)$ equal to the preceding series, $A+Bx+$, &c., and square both members, the coëfficients of the different powers of x will be the same as in the preceding solution. By equating the corresponding coëfficients, we have

$A^2=1$, whence $A=1$;

$2AB=1$, whence $B=\frac{1}{2}$;

$2AC+B^2=1$, whence $C=\frac{3}{8}$;

$2AD+2BC=0$, whence $D=-\frac{3}{16}$; &c.

$\therefore$ the series is $1+\frac{x}{2}+\frac{3x^2}{8}-\frac{3x^3}{16}+$, &c.

(8) The solution of this example is exactly like the preceding except that in equating the corresponding coëfficients the right member of each equation is 1.

(9) Since $x-x^2=x(1-x)$, let $\frac{1+x}{x-x^2}=\frac{A}{x}+\frac{B}{1-x}$.

Reducing the fractions to a common denominator, we have

$\frac{1+x}{x-x^2}=\frac{A(1-x)+Bx}{x(1-x)}$;

or, $1+x=A+(B-A)x$;

whence $A=1$, and $B-A=1$, or $B=2$.

$\therefore \frac{1+x}{x-x^2}=\frac{1}{x}+\frac{2}{1-x}$.

(10) Since $x^2-4=(x+2)(x-2)$, let $\frac{8x-4}{x^2-4}=\frac{A}{x+2}+\frac{B}{x-2}$.

$\therefore \frac{8x-4}{x^2-4}=\frac{A(x-2)+B(x+2)}{(x+2)(x-2)}=\frac{(A+B)x+(2B-2A)}{(x+2)(x-2)}$.

$8x-4=(A+B)x+(2B-2A)$;

.. $A+B=8$, and $2B-2A=-4$;

Solving these equations, we find $A=5$, and $B=3$;

$\therefore \frac{8x-4}{x^2-4}=\frac{5}{x+2}+\frac{3}{x-2}$.

(11) Since $x^2-7x+12=(x-4)(x-3)$, let $\frac{x+1}{x^2-7x+12}$

$=\frac{A}{x-4}+\frac{B}{x-3}$.

$\therefore \frac{x+1}{x^2-7x+12}=\frac{A(x-3)+B(x-4)}{(x-4)(x-3)}$

$=\frac{(A+B)x-(3A+4B)}{(x-4)(x-3)}$;

$x+1=(A+B)x-(3A+4B)$;

$\therefore$ $A+B=1$, and $-3A-4B=1$;

whence $A=5$, and $B=-4$;

$\therefore \frac{x+1}{x^2-7x+12}=\frac{5}{x-4}-\frac{4}{x-3}$.

(12) $(x^2-1)(x-2)=(x-2)(x-1)(x+1)$.

Let $\frac{x^2}{(x^2-1)(x-2)}=\frac{A}{x-2}+\frac{B}{x-1}+\frac{C}{x+1}$

$$=\frac{A(x^2-1)+B(x-2)(x+1)+C(x-2)(x-1)}{(x-2)(x-1)(x+1)}$$

$\therefore\ x^2=(A+B+C)x^2-(B+3C)x+(2C-A-2B)$;

Solving these equations, we find $A=\frac{4}{3}$, $B=-\frac{1}{2}$, $C=\frac{1}{6}$;

$$\therefore\ \frac{x^2}{(x^2-1)(x-2)}=\frac{4}{3(x-2)}-\frac{1}{2(x-1)}+\frac{1}{6(x+1)}.$$

(13) $x^4-a^4=(x^2-a^2)(x^2+a^2)=(x-a)(x+a)(x^2+a^2)$.

Let $\frac{1}{x^4-a^4}=\frac{A}{x-a}+\frac{B}{x+a}+\frac{C}{x^2+a^2}$;

$$=\frac{A(x+a)(x^2+a^2)+B(x-a)(x^2+a^2)+C(x^2-a^2)}{(x-a)(x+a)(x^2+a^2)};$$

$\therefore\ 1=(A+B)x^3+(Aa-Ba+C)x^2$
$+(Aa^2+Ba^2)x+Aa^3-Ba^3-Ca^2$.

$\therefore$ (Art. 314), $A+B=0$, (1)
$Aa-Ba+C=0$, (2)
$Aa^2+Ba^2=0$, (3)
$Aa^3-Ba^3-Ca^2=1$. (4)

Equation (3) is the same as (1); then finding the values of A, B, and C from (1), (2), and (4), we obtain

$A=\frac{1}{4a^3}$, $B=-\frac{1}{4a^3}$, and $C=-\frac{1}{2a^2}$;

$$\therefore\ \frac{1}{x^4-a^4}=\frac{1}{4a^3(x-a)}-\frac{1}{4a^3(x+a)}-\frac{1}{2a^2(x^2+a^2)}.$$

(14) $x^6-1=(x^3-1)(x^3+1)=(x-1)(x^2+x+1)(x+1)$
$(x^2-x+1)=(x-1)(x+1)(x^2-x+1)(x^2+x+1)$.

If we place $\frac{1}{x^6-1}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x^2-x+1}+\frac{D}{x^2+x+1}$, and reduce the fractions to a common denominator, and equate the coëfficients of the same powers of x, we shall find the equations incompatible, hence we must make a different assumption.

A little reflection will show that in reducing the above fractions to a common denominator, and comparing the coëfficients of the same powers of x, we shall have six independent equations, hence we may assume the numerators of the fractions so as to involve six unknown quantities, and as x may appear in the numerator of some of the fractions, we may assume it as a factor of one or more of the unknown coëfficients.

$$\therefore \text{ let } \frac{1}{x^6-1}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+C'}{x^2-x+1}+\frac{Dx+D'}{x^2+x+1}.$$

Reducing the fractions on the right to a common denominator, the numerators are

$A(x^5+x^4+x^3+x^2+x+1)+B(x^5-x^4+x^3-x^2+x-1)$
$+C(x^5+x^4-x^2-x)+C'(x^4+x^3-x-1)$
$+D(x^5-x^4+x^2-x)+D'(x^4-x^3+x-1)$, which, are $=1$.

Equating the coëfficients of the same powers of x on both sides, we have the following equations :

$$A+B+C+D=0, \quad (1)$$
$$A-B+C-D+C'+D'=0, \quad (2)$$
$$A+B+C'-D'=0, \quad (3)$$
$$A-B-C+D=0, \quad (4)$$
$$A+B-C-C'-D+D'=0, \quad (5)$$
$$A-B-C'-D'=1. \quad (6)$$

Solving these equations, we find $A=\frac{1}{6}$, $B=-\frac{1}{6}$,

$C=\frac{1}{6}$, $C'=-\frac{1}{3}$, $D=-\frac{1}{6}$, $D'=-\frac{1}{3}$.

Substituting these values, and writing $\frac{1}{6}$ as a factor of the whole, we find

$$\frac{1}{x^6-1}=\frac{1}{6}\left\{\frac{1}{x-1}-\frac{1}{x+1}+\frac{x-2}{x^2-x+1}-\frac{x+2}{x^2+x+1}\right\}.$$

BINOMIAL THEOREM,

WHEN THE EXPONENT IS FRACTIONAL OR NEGATIVE.

NOTE.— Instead of finding the general law of the coëfficients by the method given in the Algebra, page 277, it is proper to inform the student that there is another method, which is more simple in theory, but far more difficult in practice. We shall explain the method and show where the difficulty occurs.

SECOND.— To find the general law of the coëfficients.

Let $(1+x)^n=1+nx+Bx^2+Cx^3+Dx^4+$, &c., where B C, D, &c., depend upon n.

Squaring both sides, we have

$$
\begin{aligned}
(1+x)^{2n}=1+nx+Bx^2+Cx^3+Dx^4+,\ \&c.,\\
+nx+n^2x^2+Bnx^3+Cnx^4+,\ \text{``}\\
+Bx^2+Bnx^3+B^2x^4+,\ \text{``}\\
+Cx^3+Cnx^4+,\ \text{``}\\
+Dx^4+.\ \text{``}
\end{aligned}
$$

But (Art. 201), $(1+x)^{2n}=\{(1+x)^2\}^n=\{1+(2x+x^2)\}^n$.

Considering $(2x+x^2)$ as one term, we have

$$
\begin{aligned}
\{1+(2x+x^2)\}^n&=1+n(2x+x^2)+B(2x+x^2)^2+C(2x+x^2)^3+,\ \&c.\\
&=1+2nx+\ nx^2\\
&\quad +4Bx^2+4Bx^3+\ Bx^4\\
&\quad +8Cx^3+12Cx^4+,\ \&c.\\
&\quad +16Dx^4+\ \text{``}\\
&\quad +\ \text{``}
\end{aligned}
$$

Now since this series and the former must be identical, we have, by equating the coëfficients of the like powers of x,

$$2n=2n,$$

$$2B+n^2=4B+n,\ \therefore\ B=\frac{n(n-1)}{1\,.\,2};$$

$$2C+2Bn=8C+4B,\ \therefore\ C=\frac{B(n-2)}{3}=\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}$$

$$2D+2C+B^2=16D+12C+B.$$

To find D from this equation in terms of n is a difficult operation, and the finding of E would be still far more difficult. This, renders the demonstration given in the Algebra, on the whole much the easier of the two.

Article 320.

(2) Here $(n+1)\frac{b}{a+b}=(\frac{5}{2})\frac{\frac{9}{10}}{1+\frac{9}{10}}=\frac{5}{2}\times\frac{9}{10}\times\frac{10}{19}=\frac{45}{38}$; $\therefore r>1$;

hence the 2[nd] term is the greatest.

(3) Here $(n+1)\frac{b}{a+b}=(8+1)\frac{\frac{5}{2}}{3+\frac{5}{2}}=9\times\frac{5}{2}\times\frac{2}{11}=\frac{45}{11}=4\frac{1}{11}$.

The *first* whole number, greater than $4\frac{1}{11}$, is 5; therefore the 5[th] term is the greatest.

Article 321.

(1) Here $a=1, b=-x, n=-1$.

$\therefore (1-x)^{-1}=1-1\times1\times-x-\frac{1(-1-1)}{1.2}x^2$

$-\frac{1(-1-1)(-1-2)}{1.2.3}(-x)^3+$, &c.,

$=1+x+x^2+x^3+$, &c.

(2) Here $a=1, b=-x, n=-2$.

$\therefore (1-x)^{-2}=1-2\times1\times-x-\frac{2(-2-1)}{1.2}(-x)^2$

$-\frac{2(-2-1)(-2-2)}{1.2.3}(-x)^3+$, &c.,

$=1+2x+3x^2+4x^3+$, &c.

(3) To develope this expression, expand the part in the parenthesis, and multiply by a^2.

Comparing $(a+x)^{-2}$ with $(a+b)^n$, we have

$a=a, b=x$, and $n=-2$.

$\therefore (a+x)^{-2}=a^{-2}-2\times a^{-3}x-\frac{2(-2-1)}{1.2}a^{-4}x^2$

$-\frac{2(-3)(-4)}{1.2.3}a^{-5}x^3+$, &c.,

$$=\frac{1}{a^2}-\frac{2x}{a^3}+\frac{3x^2}{a^4}-\frac{4x^3}{a^5}+, \text{\&c.}$$

$$a^2(a+x)^{-2}=1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}+, \text{\&c.}$$

(4) Here $a=1, b=-x^3, n=\frac{1}{3}$.

$\therefore (1-x^3)^{\frac{1}{3}}=1-\frac{1}{3}\times1\times x^3+\frac{\frac{1}{3}(\frac{1}{3}-1)}{1.2}(-x^3)^2$

$+\frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{1.2.3}(-x^3)^3-$, &c.,

$=1-\frac{x^3}{3}-\frac{x^6}{9}-\frac{5x^9}{81}-$, &c.

(5) Here $a=a^2, b=x, n=\frac{1}{2}$.

$$\therefore\ (a^2+x)^{\frac{1}{2}}=(a^2)^{\frac{1}{2}}+\tfrac{1}{2}(a^2)^{-\frac{1}{2}}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1\,.\,2}(a^2)^{-\frac{3}{2}}x^2+$$

$$\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\,.\,2\,.\,3}(a^2)^{-\frac{5}{2}}x^3+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1\,.\,2\,.\,3\,.\,4}(a^2)^{-\frac{7}{2}}x^4+,\ \&c.,$$

$$=a+\frac{x}{2a}-\frac{x^2}{8a^3}+\frac{x^3}{16a^5}-\frac{5x^4}{128a^7}+,\ \&c.$$

In making these reductions the pupil must notice that

$$\tfrac{1}{2}(a^2)^{-\frac{1}{2}}x=\frac{x}{(2a^2)^{\frac{1}{2}}}=\frac{x}{2a},$$

$$\frac{\frac{1}{2}(\frac{1}{2}-1)}{1\,.\,2}(a^2)^{-\frac{3}{2}}x^2=-\frac{x^2}{8(a^2)^{\frac{3}{2}}}=-\frac{x^2}{8a^3},$$

$$\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\,.\,2\,.\,3}(a^2)^{-\frac{5}{2}}x^3=\frac{x^3}{16(a^2)^{\frac{5}{2}}}=\frac{x^3}{16a^5},\ \&c.$$

6) Here $a=a^3$, $b=-x$, $n=\frac{1}{3}$.

$$\therefore\ (a^3-x)^{\frac{1}{3}}=(a^3)^{\frac{1}{3}}+\tfrac{1}{3}(a^3)^{-\frac{2}{3}}\times-x+\frac{\frac{1}{3}(-\frac{2}{3})}{1\,.\,2}(a^3)^{-\frac{5}{3}}(-x)^2$$

$$+\frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{1\,.\,2\,.\,3}(a^3)^{-\frac{8}{3}}(-x)^3+\frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{1\,.\,2\,.\,3\,.\,4}(a^3)^{-\frac{11}{3}}(-x)^4$$

$$+,\ \&c.,\ =a-\frac{x}{3a^2}-\frac{x^2}{9a^5}-\frac{5x^3}{81a^8}-\frac{10x^4}{243a^{11}}-,\ \&c.$$

(7) Here $a=1$, $b=2x$, $n=\frac{1}{2}$.

$$\therefore\ (1+2x)^{\frac{1}{2}}=1+\tfrac{1}{2}(2x)+\frac{\frac{1}{2}(-\frac{1}{2})}{1\,.\,2}(2x)^2+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\,.\,2\,.\,3}(2x)^3$$

$$+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1\,.\,2\,.\,3\,.\,4}(2x)^4+,\ \&c.,$$

$$=1+x-\tfrac{1}{2}x^2+\tfrac{1}{2}x^3-\tfrac{5}{8}x^4+,\ \&c.$$

(8) Here $a=a^2$, $b=-x^2$, $n=\frac{1}{2}$.

$$\therefore\ (a^2-x^2)=(a^2)^{\frac{1}{2}}+\tfrac{1}{2}(a^2)\ ^{\frac{1}{2}}\times-x^2+\frac{\frac{1}{2}(-\frac{1}{2})}{1\,.\,2}(a^2)^{-\frac{3}{2}}(-x^2)^2$$

$$+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\,.\,2\,.\,3}(a^2)^{-\frac{5}{2}}(-x^2)^3+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1\,.\,2\,.\,3\,.\,4}(a^2)^{-\frac{7}{2}}$$

$$(\ \ x^2)^4+,\ \&c.,$$

$$=a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-\frac{5x^8}{128}-, \text{\&c.}$$

(9) $\sqrt[3]{a+x}=\sqrt[3]{\left\{a\left(1+\frac{x}{a}\right)\right\}}=\sqrt[3]{a}\sqrt[3]{\left(1+\frac{x}{a}\right)}$.

Comparing $\sqrt[3]{\left(1+\frac{x}{a}\right)}$ with $(a+b)^n$, we have $a=1$, $b=\frac{x}{a}$ $n=\frac{1}{3}$.

$$\therefore\ \sqrt[3]{\left(1+\frac{x}{a}\right)}=1+\tfrac{1}{3}\frac{x}{a}+\frac{\frac{1}{3}(-\frac{2}{3})}{1\,.\,2}\frac{x^2}{a^2}+\frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{1\,.\,2\,.\,3}\frac{x^3}{a^3}$$

$$+\frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{1\,.\,2\,.\,3\,.\,4}\frac{x^4}{a^4}+, \text{\&c.},$$

$$=1+\frac{x}{3a}-\frac{x^2}{9a^2}+\frac{5x^3}{81a^3}-\frac{10x^4}{243a^4}+, \text{\&c.}$$

$$\therefore\ \sqrt[3]{a+x}=\sqrt[3]{a}\left(1+\frac{x}{3a}-\frac{x^2}{9a^2}+\frac{5x^3}{81a^3}-\frac{10x^4}{243a^4}+, \text{\&c.}\right)$$

(10) $(a^3+x^3)^{\frac{1}{3}}=\left\{a^3\left(1+\frac{x^3}{a^3}\right)\right\}^{\frac{1}{3}}=a\left(1+\frac{x^3}{a^3}\right)^{\frac{1}{3}}$.

Comparing $\left(1+\frac{x^3}{a^3}\right)^{\frac{1}{3}}$ with $(a+b)^n$, we have $a=1$, $b=\frac{x^3}{a^3}$, $n=\frac{1}{3}$.

$$\therefore\ \left(1+\frac{x^3}{a^3}\right)^{\frac{1}{3}}=1+\tfrac{1}{3}\frac{x^3}{a^3}+\frac{\frac{1}{3}(-\frac{2}{3})}{1\,.\,2}\frac{x^6}{a^6}+\frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{1\,.\,2\,.\,3}\frac{x^9}{a^9}+, \text{\&c.}$$

$$=1+\frac{x^3}{3a^3}-\frac{2x^6}{3\,.\,6a^6}+\frac{2\,.\,5x^9}{3\,.\,6\,.\,9a^9}-, \text{\&c.}$$

$$\therefore\ (a^3+x^3)^{\frac{1}{3}}=a\left(1+\frac{x^3}{3a^3}-\frac{2x^6}{3\,.\,6a^6}+\frac{2\,.\,5x^9}{3\,.\,6\,.\,9a^9}-, \text{\&c.}\right)$$

(11) $\sqrt[3]{8+1}=\sqrt[3]{\{8(1+\frac{1}{8})\}}=2\sqrt[3]{(1+\frac{1}{8})}$. By comparing this with example 9, we find that $a=8$, $\sqrt[3]{a}=2$, and $x=1$. We may therefore obtain the development merely by substituting $\frac{1}{8}$ for $\frac{x}{a}$, in the development of $\sqrt[3]{a+x}$, or by the method pursued in the solution of that example.

(12) This is the same as example 10, except that x^3 is minus instead of plus: the development will therefore be the

same, except that the even powers of x^3, that is x^6, x^{12}, and so on, will be plus instead of minus.

(13) $(a^3-x^3)^{\frac{2}{3}}=\left\{a^3\left(1-\frac{x^3}{a^3}\right)\right\}^{\frac{2}{3}}=a^2\left(1-\frac{x^3}{a^3}\right)^{\frac{2}{3}}$;

$$\therefore\ \frac{a^3}{(a^3-x^3)^{\frac{2}{3}}}=\frac{a^3}{a^2\left(1-\frac{x^3}{a^3}\right)^{\frac{2}{3}}}=a\left(1-\frac{x^3}{a^3}\right)^{-\frac{2}{3}}.$$

Comparing $\left(1-\frac{x^3}{a^3}\right)^{-\frac{2}{3}}$ with $(a+b)^n$, we have $a=1$,

$b=-\frac{x^3}{a^3}$, $n=-\frac{2}{3}$.

$$\therefore\ \left(1-\frac{x^3}{a^3}\right)^{-\frac{2}{3}}=1-\tfrac{2}{3}\times-\frac{x^3}{a^3}-\frac{\frac{2}{3}(-\frac{2}{3}-1)}{1\,.\,2}\left(-\frac{x^3}{a^3}\right)^2$$

$$-\frac{\frac{2}{3}(-\frac{5}{3})(-\frac{8}{3})}{1\,.\,2\,.\,3}\left(-\frac{x^3}{a^3}\right)^3-\text{, \&c.,}$$

$$=1+\frac{2}{3}\cdot\frac{x^3}{a^3}+\frac{2\,.\,5x^6}{3\,.\,6a^6}+\frac{2\,.\,5\,.\,8x^9}{3\,.\,6\,.\,9a^9}+\text{, \&c.}$$

$$\therefore\ a\left(1-\frac{x^3}{a^3}\right)^{-\frac{2}{3}}=a+\frac{2}{3}\cdot\frac{x^3}{a^2}+\frac{2\,.\,5x^6}{3\,.\,6a^5}+\frac{2\,.\,5\,.\,8x^9}{3\,.\,6\,.\,9a^8}+\text{, \&c.}$$

Article 323. (16).

(1) $\sqrt{9+1}=\sqrt{9(1+\frac{1}{9})}=3\sqrt{(1+\frac{1}{9})}$; (See Formula, Art. 322.)

$$\sqrt{1+\tfrac{1}{9}}=1+\tfrac{1}{2}\times\tfrac{1}{9}-\frac{1}{2^3}\times\frac{1}{9^2}+\frac{1}{2^4}\times\frac{1}{9^3}-\frac{5}{2^6}\times\frac{1}{9^4}+\text{, \&c.,}$$

$$=1+.055555-.001543+.000085$$

$$-.000005+\text{, \&c.,}=1.054092,$$

and $1.054092\times3=3.16227+$.

(2) $\sqrt[3]{27+3}=\sqrt[3]{27(1+\frac{1}{9})}=3\sqrt[3]{1+\frac{1}{9}}$;

$$\sqrt[3]{1+\tfrac{1}{9}}=1+\tfrac{1}{3}\times\tfrac{1}{9}-\frac{1}{3^2}\times\frac{1}{9^2}+\frac{1}{3^2}\times\frac{5}{9^4}-\frac{1}{3^3}\times\frac{10}{9^5}+\text{, \&c.,}$$

$=1+.037037-.001371+.000084-.000006$
$=1.035744$; and $1.035744\times3=3.10723+$.

(3) $\sqrt[3]{27-3}=\sqrt[3]{27(1-\frac{1}{9})}=3\sqrt[3]{(1-\frac{1}{9})}$.

The development of $(1-\frac{1}{9})^{\frac{1}{3}}$ is the same as that of $(1+\frac{1}{9})^{\frac{1}{3}}$, except that all the terms after the first are negative. To get the result accurately requires that we should calculate five terms of the series after the first. These carried to nine places of decimals are

$-.037037037-.001371742-.000084675$

$-.000006272-.000000511-$, &c.

Subtracting these from 1, and multiplying the remainder by 3, we have $\sqrt[3]{24}=2.8844992+$.

(4) $\sqrt[4]{256+4}=\sqrt[4]{256(1+\frac{1}{64})}=4\sqrt[4]{1+\frac{1}{64}}$;

$\sqrt[4]{1+\frac{1}{64}}=1+\frac{1}{4}\times\frac{1}{64}-\frac{1}{4}\times\frac{3}{8}\times\frac{1}{64^2}+$, &c.,

$=1+.003906-.000022+$, &c., $=1.003884$,

and $1.003884\times4=4.01553+$.

(5) $\sqrt[7]{128-20}=\sqrt[7]{128(1-\frac{5}{32})}=2\sqrt[7]{1-\frac{5}{32}}$.

In calculating the value of each term, the shortest method is to find it from the preceding term. Thus, by considering the formula, Art. 322, we notice that each term in the development after the first, is equal to the preceding term, multiplied by two factors, one of which is $\frac{b}{a^n}$, and the others successively $\frac{1}{n}$, $\frac{n-1}{2n}$, $\frac{2n-1}{3n}$, $\frac{3n-1}{4n}$, $\frac{4n-1}{5n}$, and so on; therefore calling the terms A, B, C, and so on, we have

$\sqrt[7]{1-\frac{5}{32}}=1-\frac{1}{7}\cdot\frac{5}{32}A-\frac{3}{7}\cdot\frac{5}{32}B-\frac{13}{21}\cdot\frac{5}{32}C-\frac{5}{7}\cdot\frac{5}{32}D$
$-\frac{27}{35}\cdot\frac{5}{32}E-$, &c. $=1-.0223214-.0014947-.0001446$
$-.0000161-.0000019=.9760213$, and $.9760213\times2$
$=1.95204+$.

THE DIFFERENTIAL METHOD OF SERIES.

Article 325.

(2) Here $n=2$. $a=1$, $b=4$, $c=9$.

$\therefore D_2=1-2\times4+\frac{2(1)9}{1.2}=1-8+9=2.$

(3) Here $n=3$, $a=1$, $b=3$, $c=6$, $d=10$.

$\therefore D_3=-1+3\times3-\frac{3(2)6}{1.2}+\frac{3(2)(1)10}{1.2.3}=-1+9-18+10$

$=0.$

(4) Here $n=5$, $a=1$, $b=3$, $c=9$, $d=27$, $e=81$, $f=243$.

$\therefore D_5=-1+15-90+270-405+243=32.$

(5) Here $n=5$, $a=1$, $b=\frac{1}{2}$, $c=\frac{1}{4}$, $d=\frac{1}{8}$, $e=\frac{1}{16}$, $f=\frac{1}{32}$.

$\therefore D_5=-1+2\frac{1}{2}-\frac{5.4}{1.2}.\frac{1}{4}+\frac{5.4.3}{1.2.3}.\frac{1}{8}-\frac{5.4.3.2}{1.2.3.4}.\frac{1}{16}$

$+\frac{5.4.3.2.1}{1.2.3.4.5}.\frac{1}{32}=-1+2\frac{1}{2}-2\frac{1}{2}+1\frac{1}{4}-\frac{5}{16}+\frac{1}{32}=\frac{1}{4}-\frac{5}{16}$

$+\frac{1}{32}=-\frac{1}{32}.$

Article 326.

(3) Here $a=1$, $D_1=3$, $D_2=2$, and $D_3=0$.

$\therefore$ 15th term $=1+(15-1)3+\frac{14.13}{1.2}\times2=1+42+182$

$=225.$

n^{th} term $=1+(n-1)3+\frac{(n-1)(n-2)}{1.2}\times2=n^2.$

(4) Here $a=1$, $D_1=4$, $D_2=6$, $D_3=4$, $D_4=1$, $D_5=0$.

$\therefore$ 12th term $=1+11\times4+\frac{11.10}{1.2}\times6+\frac{11.10.9}{1.2.3}\times4$

$+\frac{11.10.9.8}{1.2.3.4}\times1=1+44+330+660+330=1365.$

(5) Here $a=1$, $D_1=2$, $D_2=1$, $D_3=0$.

$\therefore\ n^{th}$ term $=1+(n-1)2+\dfrac{(n-1)(n-2)}{1.2}=\dfrac{n^2+n}{2}$

$=\dfrac{n(n+1)}{2}$

(6) Here $a=1$, $D_1=3$, $D_2=3$, $D_3=1$, $D_4=0$.

$\therefore\ n^{th}$ term $=1+(n-1)3+\dfrac{(n-1)(n-2)}{1.2}\times 3$

$+\dfrac{(n-1)(n-2)(n-3)}{1.2.3}\times 1$

$=\dfrac{6+18n-18+9n^2-27n+18+n^3-6n^2+11n-6}{1.2.3}$

$=\dfrac{n^3+3n^2+2n}{1.2.3}=\dfrac{n(n^2+3n+2)}{1.2.3}=\dfrac{n(n+1)(n+2)}{1.2.3}$.

(7) Multiplying the factors together, the terms are 70, 252 594, 1144, 1950, and so on.
Here $a=70$, $D_1=182$, $D_2=160$, $D_3=48$, $D_4=0$.

$\therefore\ 9^{th}$ term $=70+8\times 182+\dfrac{8\times 7}{1.2}\times 160+\dfrac{8.7.6}{1.2.3}\times 48$

$=70+1456+4480+2688=8694$.

(8) Here the terms are 2, 12, 30, 56, and so on;
hence $a=2$, $D_1=10$, $D_2=8$, $D_3=0$.

$\therefore\ n^{th}$ term $=2+(n-1)10+\dfrac{(n-1)(n-2)}{2}\times 8$

$=2+10n-10+4n^2-12n+8=4n^2-2n$.

Article 327.

(3) Here $a=1$, $D_1=2$, $D_2=1$, $D_3=0$.

$\therefore$ Sum of n terms $=n+\dfrac{n(n-1)}{1.2}\times 2+\dfrac{n(n-1)(n-2)}{1.2.3}\times 1$

$=\dfrac{6n+6n^2-6n+n^3-3n^2+2n}{1.2.3}=\dfrac{n^3+3n^2+2n}{1.2.3}$

$=\dfrac{n(n+1)(n+2)}{1.2.3}$.

(4) Here $a=3$, $D_1=8$, $D_2=12$, $D_3=6$, $D_4=0$.

$\therefore$ Sum of 20 terms $=20\times3+\frac{20\,.\,19}{1\,.\,2}\times8+\frac{20\,.\,19\,.\,18}{1\,.\,2\,.\,3}$

$\times12+\frac{20\times19\times18\times17}{1\,.\,2\,.\,3\,.\,4}\times6=60+1520+13680+29070$

$=44330.$

(5) Here the terms are 6, 24, 60, 120, 210, and so on; hence $a=6$, $D_1=18$, $D_2=18$, $D_3=6$, and $D_4=0$.

$\therefore$ Sum of 20 terms $=20\times6+\frac{20\times19}{1\,.\,2}\times18$

$+\frac{20\times19\times18}{1\,.\,2\,.\,3}\times18+\frac{20\times19\times18\times17}{1\,.\,2\,.\,3\,.\,4}\times6=120+3420$

$+20520+29070=53130.$

(6) Here $a=1$, $D_1=7$, $D_2=12$, $D_3=6$, $D_4=0$.

$\therefore$ Sum of n terms $=n+\frac{n(n-1)}{1\,.\,2}\times7+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}$

$\times12+\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}\times6=\frac{4n}{4}+\frac{14n^2-14n}{4}$

$+\frac{8n^3-24n^2+16n}{4}+\frac{n^4-6n^3+11n^2-6n}{4}$

$=\frac{n^4+2n^3+n^2}{4}=\frac{n^2}{4}(n^2+2n+1)=[\frac{1}{2}n(n+1)]^2.$

(7) Here $a=1$, $D_1=3$, $D_2=3$, $D_3=1$, $D_4=0$.

$\therefore$ Sum of n terms $=n+\frac{n(n-1)}{1\,.\,2}\times3+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\times3$

$+\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}=\frac{24n}{24}+\frac{36n^2-36n}{24}$

$+\frac{12n^3-36n^2+24n}{24}+\frac{n^4-6n^3+11n^2-6n}{24}$

$=\frac{n^4+6n^3+11n^2+6n}{24}=\frac{n(n^3+6n^2+11n+6)}{24}$

$=\frac{n(n+1)(n+2)(n+3)}{24}.$

As the direct method of factoring $n^4+6n^3+11n^2+6n$, is a work of some difficulty to the learner, we shall explain the operation.

It is evident from Articles 234 and 253, and the principle is proved directly in Art. 395, that if we put $n^4+6n^3+11n^2+6n=0$ and find the values of n, which we may suppose to be a, b, c, d, then the factors of the expression will be $n-a, n-b, n-c, n-d$

By proceeding to solve the equation

$$n^4+6n^3+11n^2+6n=0,$$

according to the method explained in Art. 243, we have $(n^2+3n)^2+2(n^2+3n)=0$; from this equation we find $n^2+3n=0$, or -2. From the equation $n^2+3n=0$, we have $n=0$, or -3; and from the equation $n^2+3n=-2$, we have $n=-1$, or -2; $\therefore$ the factors required are n, $n+1$, $n+2$, and $n+3$.

(8) We must find this series by making $n=1, 2, 3$, and so on; thus, if $n=1$, the first term is 1; if $n=2$ the second term is 16; if $n=3$, the third term is 63; in like manner we find the fourth term is 160, the fifth term 325, and so on. Hence, we find $a=1$, $D_1=15$, $D_2=32$, $D_3=18$, $D_4=0$.

$$\therefore \text{ Sum of 25 terms } =25+\frac{25 \cdot 24}{1 \cdot 2}\times 15+\frac{25 \cdot 24 \cdot 23}{1 \cdot 2 \cdot 3}\times 32$$

$$+\frac{25 \cdot 24 \cdot 23 \cdot 22}{1 \cdot 2 \cdot 3 \cdot 4}\times 18=25+4500+73600+227700$$

$$=305825.$$

PILING OF CANNON BALLS AND SHELLS.

Articles 328—332.

(3) Comparing the number 15 with Formula B in Art. 332, we have $n=15$.

$$\therefore \text{ number } =\frac{n(n+1)(2n+1)}{6}=\frac{15\times 16\times 31}{6}=1240.$$

(4) See Formula C, Art. 332, $l=52$, and $n=34\frac{1}{6}$;

$n(n+1)(3l-n+1)=\frac{34}{6}\times 35\times(156-34+1)=\frac{17}{3}\times 35 \times 123=17\times 35\times 41=24395.$

(5) Number of balls in a complete triangular pile of which each side of the base is 25, is (Art. 332),

$\frac{1}{6}n(n+1)(n+2)=\frac{25}{6}\times 26\times 27=25\times 13\times 9=2925.$

Since the number of balls in a side of the top course is

13, the number in a side of the pile that is wanting is 12, hence the number in this pile is $\frac{12}{6}\times.3\times14=364$.
$\therefore$ 2925—364=2561, the number required.

(6) Number in the pile considered as complete, (Art. 332),

$$=\frac{n}{6}(n+1)(n+2)=\frac{38}{6}\times39\times40=19\times13\times40=9880.$$

Since there are 15 courses, and the number of balls is one less in each course than in the next preceding course therefore 38—15=23 is the number of balls in a side of the incomplete pile, and the number in this pile is

$\frac{23}{6}\times24\times25=23\times4\times25=2300$.
$\therefore$ 9880—2300=7580, the number required.

(7) Number in the pile considered as complete (Art. 332),

$$=\frac{n}{6}(n+1)(2n+1)=\frac{44}{6}\times45\times89=22\times15\times89=29370.$$

Number of balls in a side of the pile that is wanting is 21, and the number in the incomplete pile is

$\frac{21}{6}\times22\times43=7\times11\times43=3311$.
$\therefore$ 29370—3311=26059, the number in the incomplete square pile.

(8) $\sqrt{1521}=39=$ number of balls in a side of the base course,
$\sqrt{169}=13=$ " " " " " top "
$\frac{39}{6}\times40\times79=13\times20\times79=20540$, the number of balls in the pile considered as complete.
13—1=12, the number of balls in a side of the base of the pile that is wanting; and $\frac{12}{6}\times13\times25=650$.
$\therefore$ 20540—650=19890, the number of balls in the incomplete pile.

(9) Here we have the equation (Art. 332),
$\frac{1}{6}n(n+1)(3l-n+1)=6440$,
in which $n=20$, to find l.
$\therefore$ $\frac{20}{6}\times21(3l-19)=6440$,
$70(3l-19)=6440$,
$3l-19=92$, and $l=37$.
$37\times20=740$, the number of balls in the base.

(10) Here we have the proportion

$\frac{1}{6}n(n+1)(n+2) : \frac{1}{6}n(n+1)(2n+1) :: 6 : 11.$

Placing the product of the means equal to the product of the extremes, and canceling $\frac{1}{6}n(n+1)$ on each side, we have $12n+6=11n+22$,

whence $n=16$, the number of balls in a side of the base of each.

$\frac{1}{6}n(n+1)(n+2)=\frac{16}{6}\times17\times18=816$ = balls in tr. pile,
$\frac{1}{6}n(n+1)(2n+1)=\frac{16}{6}\times17\times33=1496=$ " " sq. pile.

(11) Since the number of balls in each side increases by 1 as we descend, and since there are 7 courses below the upper one, therefore $36+7=43$, and $17+7=24$, are the number of balls in the longer and shorter sides of the lower course,

$\frac{1}{6}n(n+1)(3l-n+1)=\frac{24}{6}\times25(129-24+1)=10600,$

the number of balls in the pile considered as complete. It is evident that 35 and 16 are the number of balls in the longer and shorter sides of the pile that is wanting, hence the number of balls in this pile, is

$\frac{16}{6}\times17(105-16+1)=4080.$

$\therefore$ $10600-4080=6520$, the number of balls in the incomplete pile.

INTERPOLATION OF SERIES.

Article 333—335.

(1) Since the 4^{th} differences vanish, we have (Art. 325),

$e-4d+6c-4b+a=0$, where $a=3$, $c=15$
$d=30$, and $e=55$, to find b.

$\therefore$ $55-4\times30+6\times15-4b+3=0,$

whence $b=7$.

Having the terms of the series, viz. . 3, 7, 15, 30, 55, we readily find the first terms of the several orders of differences (Art. 325) to be $D_1=4$, $D_2=4$, $D_3=3$, and $D_4=0$, therefore by making $n=6$, 7, and 8 successively, and substituting the values of D_1, D_2, and D_3 in the formula

$$a+(n-1)D_1+\frac{(n-1)(n-2)}{1.2}D_2+\frac{(n-1)(n-2)(n-3)}{1.2.3}D_3,$$

we obtain the 6^{th}, 7^{th}, and 8^{th} terms. Thus the 6^{th} term is

$$3+5\times4+5\times4\times2+\frac{5\times4\times3}{1\times2\times3}\times3$$

$=3+20+40+30=93.$

7^{th} term is $3+6\times4+6\times5\times2+\frac{6\times5\times4}{1\times2\times3}\times3=3+24+60$ $+60=147.$

8^{th} term is $3+7\times4+7\times6\times2+\frac{7\times6\times5}{1\times2\times3}\times3=3+28+84$ $+105=220.$

(2) Let $x=$ the 5^{th} term, then writing down the terms, and finding the respective orders of differences, we have

$$\begin{array}{l} 11\ ,\ 18\ ,\ 30\ ,\ 50\ ,\ x\ ,\ 132\ ,\ 209, \\ \quad 7\ ,\ 12\ ,\ 20\ ,\ x-50\ ,\ 132-x\ ,\ 77\ , \\ \quad\quad 5\ ,\ 8\ ,\ x-70\ ,\ 182-2x\ ,\ x-55\ , \\ \quad\quad\quad 3\ ,\ x-78\ ,\ 252-3x,\ 3x-237, \\ \quad\quad\quad\quad x-81\ ,\ 330-4x,\ 6x-489, \\ \quad\quad\quad\quad\quad 411-5x\ ,\ 10x-819, \\ \quad\quad\quad\quad\quad\quad 15x-1230. \end{array}$$

$\therefore$ $15x-1230=0$, and $x=82$.

Or, thus, since the 6^{th} differences vanish, or become 0, it is merely necessary to find the first term of the 6^{th} order, by means of the Formula, Art. 325, by calling $n=6$, $a=11$, $b=18$, &c., thus,

$$a-nb+\frac{n(n-1)}{1\,.\,2}c-\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}d+\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}x$$

$$-\frac{n(n-1)(n-2)(n-3)(n-4)}{1\,.\,2\,.\,3\,.\,4\,.\,5}f$$

$$+\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6}g=0\ ;$$

$$\therefore\ 11-6\times18+\frac{6\times5}{2}\times30-\frac{6\times5\times4}{2\times3}\times50+\frac{6\times5\times4\times3}{2\times3\times4}x$$

$$-\frac{6\times5\times4\times3\times2}{2\times3\times4\times5}\times132+\frac{6\times5\times4\times3\times2\times1}{1\times2\times3\times4\times5\times6}\times209=0\ ;$$

$11-108+450-1000+15x-792+209=0,$

whence $15x=1230$, and $x=82$.

(3) Calling the respective given logarithms, a, b, d, and e, since c is wanting, we have, by the formula, Art. 325,
$e-4d+6c-4b+a=0$.
From this equation, by substituting the values of a, b, d, and e, we readily find $c=2.0128372$.

(4)

Nos.	Cube Roots.	1st Diff.	2nd Diff.	Mean of 2nd Diff.
60	3.91487			
62	3.95789	4302		
		4211	—91	—89
64	4.		—87	
66	4.04124	4124		

Here $t=\frac{1}{2}$, $d=4211$, $d'=-89$, (Art. 335), and

$$t\left(d+\frac{t-1}{2}d'\right)=\tfrac{1}{2}\left(4211+\frac{\frac{1}{2}-1}{2}\times-89\right)=\tfrac{1}{2}(4211+22)$$

$=2116$; and $3.95789+.02116=3.97905$.

(5) Let a^2, $(a+1)^2$, $(a+2)^2$, &c., be a series of squares, and let them be developed, and their differences be taken as below:

$$\begin{array}{cccc} a^2, & a^2+2a+1, & a^2+4a+1, & a^2+6a+9, \text{ \&c.,} \\ & 2a+1, & 2a+3, & 2a+5, \text{ \&c.,} \\ & & 2, & 2, \text{ \&c.} \end{array}$$

The second differences are constant, and a table of squares may be found as follows:

Let us commence with $50^2=2500$, and $51^2=2601$; whose difference is 101; then since the second differences are constant and equal to 2, the difference between the squares of 51 and 52 will be $101+2=103$, and this added to 2601 will give the square of 52; and so on, as in the following table:

$2500=50^2$	$2809=53^2$
101	107
$2601=51^2$	$2916=54^2$
103	109
$2704=52^2$	$3025=55^2$
105	111
$2809=53^2$	$3136=56^2$.

In a manner nearly similar, a table of cube numbers may be computed.

INFINITE SERIES.

Articles 336—338.

(2) Since $q=1, p=1$, and $n=1, 2, 3$, &c.

$$\therefore \left\{\begin{matrix} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+, \text{\&c., ad inf.} \\ -(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+, \text{\&c.,) ad inf.} \end{matrix}\right\} =1= \text{sum}$$

(3)
$$\left\{\begin{matrix} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \cdot \cdot \cdot \cdot \frac{1}{n} \\ -\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \cdot \cdot \frac{1}{n}+\frac{1}{n+1}\right) \end{matrix}\right\} =1-\frac{1}{n+1}=\frac{n}{n+1}.$$

(4) Since $q=1$, and $p=3$,

$$\therefore \left\{\begin{matrix} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+, \text{\&c.,} \\ -(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+, \text{\&c.)} \end{matrix}\right\} =1+\frac{1}{2}+\frac{1}{3}=1\frac{5}{6},$$

and $\frac{1}{p}^{th}$ of this sum $=\frac{11}{18}=$ sum required

(5) Since $q=1, p=2$, and $n=1, 2, 3$, &c.

$$\therefore \left\{\begin{matrix} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+, \text{\&c.,} \\ -(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+, \text{\&c.)} \end{matrix}\right\} =1+\frac{1}{2}=\frac{3}{2},$$

and $\frac{1}{p}^{th}$ of this sum $=\frac{1}{2}$ of $\frac{3}{2}=\frac{3}{4}$.

(6) To find the series let $n=1, 2, 3, 4$, &c., successively, then the terms are

$$\frac{1}{1.5}+\frac{1}{2.6}+\frac{1}{3.7}+\frac{1}{4.8}+, \text{\&c.}$$

Also, $q=1$, and $p=4$.

$$\therefore \left\{\begin{matrix} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+, \text{\&c.,} \\ -(\frac{1}{5}+\frac{1}{6}+, \text{\&c.)} \end{matrix}\right\} =1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{25}{12}$$

and $\frac{1}{p}^{th}$ of this sum $=\frac{1}{4}$ of $\frac{25}{12}=\frac{25}{48}$.

(7) Dividing each term of this series by 2, it becomes

$$\frac{1}{2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+, \text{\&c.}$$

The sum of this series has been found (see example 2.) to be 1 therefore the sum of the given series is $1\times2=2$.

(8) Multiplying each term of this series by 3×4, or 12, it becomes $\frac{1}{2}+\frac{1}{2.3}+\frac{1}{3.4}+$, &c., the sum of which has been found to be 1; therefore the sum of the series is $1\div12=\frac{1}{12}$.

RECURRING SERIES.

Articles 339—343.

(2) Here $A=1$, $B=6x$, $C=12x^2$, $D=48x^3$, $E=120x^4$, &c. Making $x=1$, and substituting in the formula, (Art. 341), we have $p=\frac{12\times48-6\times120}{12\times12-6\times48}=1$, $q=\frac{12\times120-48\times48}{12\times12-6\times48}=6$.

(Art. 343), $S=\frac{A+B-Apx}{1-px-qx^2}=\frac{1+6x-x}{1-x-6x^2}=\frac{1+5x}{1-x-6x^2}$.

(3) Here $A=1$, $B=2x$, $C=3x^2$, $D=4x^3$, $E=5x^4$. Making $x=1$, and applying formula (Art. 341), we have $p=\frac{3\times4-2\times5}{3\times3-2\times4}=2$, $q=\frac{3\times5-4\times4}{3\times3-2\times4}=-1$.

(Art. 343), $S=\frac{A+B-Apx}{1-px-qx^2}=\frac{1+2x-2x}{1-2x+x^2}=\frac{1}{(1-x)^2}$.

(4) Here $A=\frac{a}{c}$, $B=-\frac{abx}{c}$, $C=\frac{ab^2x^2}{c^3}$, $D=-\frac{ab^3x^3}{c^4}$, &c.

If we make $x=1$, and apply the formula, (Art. 341), we shall find $p=-\frac{b}{c}$, and $q=0$, but the scale of relation is easily seen to be $-\frac{b}{c}$, since if any coëfficient is multiplied by this quantity it will give the coëfficient of the next following term.

(Art. 343), $S=\frac{A+B-Apx}{1-px}=\frac{\frac{a}{c}-\frac{abx}{c^2}+\frac{abx}{c^2}}{1+\frac{bx}{c}}=\frac{a}{c+bx}$.

(5) He e $A=0$, $B=x$, $C=x^2$, &c., and the scale of relation is that is $p=1$, and $q=0$.

$$\therefore \ S=\frac{A+B-Apx}{1-px}=\frac{0+x-0}{1-x}=\frac{x}{1-x}.$$

(6) Here $A=0$, $B=x$, $C=-x^2$, &c., and the scale of relation is -1, that is $p=-1$, and $q=0$.

$$\therefore \ S=\frac{A+B-Apx}{1-px}=\frac{0+x-0}{1+x}=\frac{x}{1+x}.$$

(7) Here $A=1$, $B=2x$, $C=8x^2$, $D=28x^3$, $E=100x^4$.
Making $x=1$, and applying formula, (Art. 341), we have

$$p=\frac{8\times28-2\times100}{8\times8-2\times28}=3,\ q=\frac{8\times100-28\times28}{8\times8-2\times28}=2.$$

(Art. 343), $S=\dfrac{A+B-Apx}{1-px-qx^2}=\dfrac{1+2x-3x}{1-3x-2x^2}=\dfrac{1-x}{1-3x-2x^2}.$

(8) Here $A=1$, $B=3x$, $C=5x^2$, $D=7x^3$, $E=9x^4+$, &c.
Making $x=1$, and applying the formula, (Art. 341), we have

$$p=\frac{5\times7-3\times9}{5\times5-3\times7}=2,\ q=\frac{5\times9-7\times7}{5\times5-3\times7}=-1.$$

(Art. 343), $S=\dfrac{A+B-Apx}{1-px-qx^2}=\dfrac{1+3x-2x}{1-2x+x^2}=\dfrac{1+x}{(1-x)^2}.$

(9) Here $A=1$, $B=4x$, $C=9x^2$, $D=16x^3$, $E=25x^4$, &c.
Making $x=1$, and applying the formula, (Art. 341.)
The values of p and q thus found will not reproduce the series, hence we must apply the equations in Art. 342, and find the values of p, q, and r, when $x=1$. These equations give

$$16=9p+4q+r;$$
$$25=16p+9q+4r;$$
$$36=25p+16q+9r.$$

From these equations we find $p=3$, $q=-3$, and $r=1$.

We shall now extend the principle of Art. 343 to finding the sum of an infinite recurring series when the scale of relation consists of three terms.

The 1st term		$A=A$;
the 2nd	"	$B=B$;
the 3rd	"	$C=C$;
the 4th	"	$D=Cpx+Bqx^2+Arx^3$;
the 5th	"	$E=Dpx+Cqx^2+Brx^3$;
the 6th	"	$F=Epx+Dqx^2+Crx^3$;

&c., = &c.

Now if S represents the required sum, by adding together the corresponding members of these equalities, and observing that C+D+E+, &c., =S—A—B ; B+C+D+, &c., =S—A, we have

$$S=A+B+C+(S-A-B)px+(S-A)qx^2+Srx^3,$$

whence $S=\dfrac{A+B+C-(A+B)px-Aqx^2}{1-px-qx^2-rx^3}$.

Substituting in this formula the values of A, B, C, and of p, q, and r; we have

$$S=\frac{1+4x+9x^2-3x-12x^2+3x^2}{1-3x+3x^2-x^3}=\frac{1+x}{(1-x)^3}.$$

REVERSION OF SERIES.

Articles 344—346.

(1) Comparing the series with the formula, (Art. 344), we have $a=+1$, $b=-1$, $c=+1$, $d=-1$, &c.,
hence by substitution, we have

$$x=\frac{1}{1}y-\frac{-1}{1}y^2+\frac{2-1}{1}y^3-\frac{-1+5-5}{1}y^4+, \&c.,$$
$$=y+y^2+y^3+y^4+, \&c.$$

(2) Here $a=1$, $b=1$, $c=1$, $d=1$, &c.

$$\therefore\ x=\frac{1}{1}y-\frac{1}{1}y^2+\frac{2-1}{1}y^3-\frac{1-5+5}{1}y^4+, \&c.,$$
$$=y-y^2+y^3-y^4+, \&c.$$

(3) Comparing the coëfficients with those of the series in Art. 346, we have $a=2$, $b=3$, $c=4$, $d=5$, &c.

$$\therefore\ x=\frac{1}{2}y-\frac{3}{16}y^3+\frac{27-8}{128}y^5-, \&c.,$$
$$=\tfrac{1}{2}y-\tfrac{3}{16}y^3+\tfrac{19}{128}y^5-, \&c.$$

(4) Applying the formula, (Art. 345), we have
$a'=1$, $a=-2$, and $b=3$.

$$\therefore\ x=-\frac{1}{2}(y-1)+\frac{3}{8}(y-1)^2-\frac{18-0}{32}(y-1)^3+, \&c.,$$
$$=-\tfrac{1}{2}(y-1)+\tfrac{3}{8}(y-1)^2-\tfrac{9}{16}(y-1)^3+, \&c.$$

(5) See formula, Art. 345. Here $a'=1$, $a=1$, $b=-2$, $c=+$

$\therefore\ x=\frac{1}{1}(y-1)+\frac{2}{1}(y-1)^2+\frac{8-1}{1}(y-1)^3-\frac{0+10-40}{1}$
$(y-1)^4$,
$=y-1+2(y-1)^2+7(y-1)^3+30(y-1)^4+$, &c.

(6) See formula, Art. 344. Here $a=1$, $b=\frac{1}{2}$, $c=\frac{1}{6}$, $d=\frac{1}{24}$, &c.

$$\therefore\ x=\tfrac{1}{1}y-\frac{\frac{1}{2}}{1}y^2+\frac{\frac{1}{2}-\frac{1}{6}}{1}y^3-\frac{\frac{1}{24}-\frac{5}{12}+\frac{5}{8}}{1}y^4+,\ \&c.,$$

$$=y-\tfrac{1}{2}y^2+\tfrac{1}{3}y^3-\tfrac{1}{4}y^4+,\ \&c.$$

(7) Let $x=Ay+By^2+Cy^3+$, &c.;
then $x^2=A^2y^2+2ABy^3+$, &c.;
$x^3=\qquad A^3y^3+$, &c.

Substituting these values for $x, x^2, x^3, \ldots$ in the second member of the given equation, and transposing the first member, we have

$$0=\left.\begin{matrix}gA\\-1\end{matrix}\right|y+\left.\begin{matrix}gB\\hA^2\\-a\end{matrix}\right|y^2+\left.\begin{matrix}gC\\+2hAB\\+\ kA^3\\-b\end{matrix}\right|y^3+\ldots$$

Hence $Ag-1=0$, $A^2h+Bg-a=0$,
$A^3k+2ABh+gC-b=0$;

whence, $A=\frac{1}{g}$, $B=\frac{1}{g}(a-A^2h)=\frac{1}{g}\left(a-\frac{h}{g^2}\right)=\frac{1}{g^3}(ag^2-h)$

$$C=\frac{1}{g}(b-A^3k-2ABh)$$
$$=\frac{1}{g}\left\{b-\frac{k}{g^3}-\frac{2}{g}\cdot\frac{1}{g^3}(ag^2-h)h\right\}$$
$$=\frac{bg^4-kg-2h(ag^2-h)}{g^5},$$

$$\therefore\ x=\frac{y}{g}+\frac{(ag^2-h)y^2}{g^3}+\frac{[bg^4-kg-2h(ag^2-h)]y^3}{g^5}+,\ \&c.$$

CONTINUED FRACTIONS; LOGARITHMS; EXPONENTIAL EQUATIONS; INTEREST AND ANNUITIES.

CONTINUED FRACTIONS.

Articles 347—356.

(1) Dividing the greater term by the less, the last divisor by the last remainder, and so on, the quotients are 3, 4, 5, and 6; hence the integral fractions are $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$, and the converging fractions are

$$\frac{1}{3},\ \frac{1\times 4}{3\times 4+1}=\frac{4}{13},\ \frac{4\times 5+1}{13\times 5+3}=\frac{21}{68},\ \frac{21\times 6+4}{68\times 6+13}=\frac{130}{421}.$$

The 2nd and 3rd examples are worked in a similar manner.

(4) Making 3900 the numerator, and 10963 the denominator of a fraction, and proceeding as in the preceding examples, the successive quotients, that is the denominators of the respective integral fractions, are 2, 1, 4, 3, 2, 2, 1, 30; hence the first approximate fraction is $\frac{1}{2}$, the second, $\frac{1\times 1}{2\times 1+1}=\frac{1}{3}$; the third, $\frac{1\times 4+1}{3\times 4+2}=\frac{5}{14}$; and so on.

(5) Making 4900 the numerator, and 11283 the denominator of a fraction, and proceeding as above, we find the successive quotients to be 2, 3, 3, 3, 2, 7, 1, 1, 1, 2; hence the approximating fractions are $\frac{1}{2}$; $\frac{1\times 3}{2\times 3+1}=\frac{3}{7}$;

$$\frac{3\times 3+1}{7\times 3+2}=\frac{10}{23};\ \frac{10\times 3+3}{23\times 3+7}=\frac{33}{76};\ \frac{33\times 2+10}{76\times 2+23}=\frac{76}{175},\ \&c.$$

(6) Making 1 the numerator, and 3.1415926 the denominator of a fraction, or 10000000 and 31415926, and dividing the greater by the less, the less by the remainder, and so on, the quotients are 3, 7, 15, 1, 243, &c. Operating in a similar manner with 1, and 3.1415927, the quotients are 3, 7, 15, 1, 354, &c., then finding the approximating or converging fractions, corresponding to these quotients, we have $\frac{1}{3}$; $\frac{1\times 7}{3\times 7+1}=\frac{7}{22}$; $\frac{7\times 15+1}{22\times 15+3}=\frac{106}{333}$; $\frac{106\times 1+7}{333\times 1+22}$ $=\frac{113}{355}$.

The ratio of 113 to 355, that is $\frac{355}{113}=3.1415929+$; and since the true ratio lies between 3.141526, and 3.1415927, and since the difference between 3.1415929 and 3.1415926 is .0000003, therefore $\frac{113}{355}$ expresses the part that the diameter is of the circumference to within less than .0000003.

(7) 5 hrs, 48 min., 49 sec., =20929 seconds,

24 hrs, = 86400 "

Operating with these numbers as before, we find the successive quotients to be

4, 7, 1, 3, 1, 16, 1, 1, 15; and from these the converging fractions are readily found.

(8) Dividing the greater term by the less, the less by the remainder, and so on, the quotients are

1, 1, 2, 1, 1, 1, 3, 2, 1, 1, 2, 3; and the successive converging fractions found from these are $\frac{1}{1}$, $\frac{1}{2}$, $\frac{3}{5}$, $\frac{4}{7}$, $\frac{7}{12}$, $\frac{11}{19}$, $\frac{40}{69}$, &c., whence $\frac{11}{19}$ is the required fraction.

(9) In solving this example it is most convenient to consider 1 as the numerator, and 27.321661 the denominator, and then invert the resulting converging fractions. Dividing 27.321661 by 1, or 27321661 by 1000000, as in the preceding examples, the quotients are

27, 3, 9, 5, 2, &c.; these give for approximating fractions $\frac{1}{27}$, $\frac{3}{82}$, $\frac{28}{765}$, $\frac{143}{3907}$, &c., hence the required ratios are $\frac{27}{1}$, $\frac{82}{3}$, $\frac{765}{28}$, $\frac{3907}{143}$, &c.

(10) Referring to Art. 353, we have $a=1$,

$$\text{hence } \sqrt{2}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+}}}, \text{ \&c.}$$

The integral fractions are $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, &c.,

" converging fractions are $\frac{1}{2}$, $\frac{2}{5}$, $\frac{5}{12}$, $\frac{12}{29}$, &c.

Adding 1 to each of these we have $\frac{3}{2}$, $\frac{7}{5}$, $\frac{17}{12}$ $\frac{41}{29}$, &c

(11) Referring to Art. 353, we have $a=2$, and $2a=4$,

hence $\sqrt{4+1}=2+\cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{4+}}}}$, &c.

The int. fractions are $\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \ \frac{1}{4}$, &c.

" conv. fractions are $\frac{1}{4}, \frac{4}{17}, \frac{17}{72} \ \frac{72}{305}, \frac{305}{1292}$, &c.

Adding 2 to each of these, we have $\frac{9}{4}, \frac{38}{17}, \frac{161}{72}, \frac{682}{305}, \frac{2889}{1292}$, &c.

Now the fourth fraction being in an even place is *less* than the true value, and the fifth being in an odd place is *greater* than the true value, therefore $\sqrt{5}$ is greater than $\frac{682}{305}$, and less than $\frac{2889}{1292}$

(12) Since $8^1=8$, and $8^2=64$, x lies between 1 and 2

hence let $x=1+\frac{1}{y}$.

$\therefore\ 8^{1+\frac{1}{y}}=32$, or $8\times 8^{\frac{1}{y}}=32$,

or, $8^{\frac{1}{y}}=\frac{32}{8}=4$.

or, $8=4^y$, by raising both members to the y power,

Now since $4^1=4$, and $4^2=16$, the value of y lies between 1 and 2, hence let $y=1+\frac{1}{z}$.

$\therefore\ 4^{1+\frac{1}{z}}=8$, or $4\times 4^{\frac{1}{z}}=8$, or $4^{\frac{1}{z}}=2$;

raising both members to the z power, we have

$2^z=4$, whence $z=2$.

$\therefore\ x=1+\cfrac{1}{1+\cfrac{1}{2}}=1+\frac{2}{3}=\frac{5}{3}$.

(13) $3^x=25$; $3^2=9$, and $3^3=27$.

$\therefore\ x=2+\frac{1}{x'}$.

$3^{2+\frac{1}{x'}}=15$, or $3^2\times 3^{\frac{1}{x'}}=15$, or $3^{\frac{1}{x'}}=\frac{15}{9}=\frac{5}{3}$.

Since $3^{\frac{1}{x'}}=\frac{5}{3}$, we have $(\frac{5}{3})^{x'}=3$;

here $x'=2+\frac{1}{x''}$,

$\therefore (\frac{5}{3})^{2+\frac{1}{x''}}=3$, or $(\frac{5}{3})^2\times(\frac{5}{3})^{\frac{1}{x''}}=3$,

whence $(\frac{5}{3})^{\frac{1}{x''}}=\frac{27}{25}$,

or, $(\frac{27}{25})^{x''}=\frac{5}{3}$;

here $x''=6+\frac{1}{x'''}$.

$$\therefore\ x=2+\cfrac{1}{2+\cfrac{1}{6+,\&c.}}$$

hence the approximating fraction to be added to 2, is $\frac{1}{2}$, or $\frac{6}{13}$;

$\frac{6}{13}=2.46+$, which is true to within $(\frac{1}{13})^2=\frac{1}{169}$.

This method of finding the value of x is more curious than useful, as the same thing may be accomplished directly, and with but little labor, by means of logarithms.

$$\text{Thus } x=\frac{\text{log. } 15}{\text{log. } 3}=\frac{1.1760913}{.4771213}=2.465 \text{ nearly.}$$

LOGARITHMS.

Article 366.

(1) The result in this example follows directly from Art. 360, the pupil, however, may prove the principle generally in the case of three factors; thus,

$a^x=\text{N}$ . . . . (1),
$a^{x'}=\text{N}'$ . . . . (2),
$a^{x''}=\text{N}''$. . . . . . . (3).

Multiplying equations (1), (2), and (3) together, we have $a^x\times a^{x'}\times a^{x''}=a^{x+x'+x''}=\text{NN}'\text{N}''$.

But, by the definition of logarithms, if we consider a the base of the system, then x, x', and x'' are the logarithms of N, N′, and N″, and $(x+x'+x'')$ is the logarithm of NN′N″, hence, the sum of the logarithms of three numbers is equal to the logarithm of their product.

(2) By Art. 361, log. $\left(\frac{abc}{de}\right)=$ log. $(abc)-$ log. (de); but

log. $(abc)=$ log. $a+$ log. $b+$ log. c; and log. $(de)=$ log. d + log. e; hence log. $\left(\frac{abc}{de}\right)=$ log. $a+$ log. $b+$ log. c $-$(log. $d+$ log. e)= log. $a+$ log. $b+$ log. $c-$ log. $d-$ log. e.

(3) By Art. 360, log. $(a^m \cdot b^n \cdot c^p.)=$ log. a^m+ log. b^n+ log. c^p; but (Art. 362), log. $a^m=m$ log. a, log. $b^n=n$ log. b, and log. $c^p=p$ log. c.
$\therefore$ log. $(a^m \cdot b^n \cdot c^p)=m$ log. $a+n$ log. $b+p$ log. c.

(4) Log. $\left(\frac{a^m \cdot b^n}{c^p}\right)=$ log. $(a^m \cdot b^n)-$ log. $c^p=m$ log. $a+$ n log. $b-p$ log. c.

(5) $a^2-x^2=(a+x)(a-x)$, and log. (a^2-x^2) $=$ log. $\{(a+x)(a-x)\}=$ log. $(a+x)+$ log. $(a-x)$.

(6) Since log. $(a^2-x^2)=$ log. $(a+x)+$ log. $(a-x)$,
$\frac{1}{2}$ log. $(a^2-x^2)=\frac{1}{2}$ log. $(a+x)+\frac{1}{2}$ log. $(a-x)$,
but, (Art. 363), $\frac{1}{2}$ log. $(a^2-x^2)=$ log. $(a^2-x^2)^{\frac{1}{2}}$,
or, log. $\sqrt{a^2-x^2}$;
$\therefore$ log. $\sqrt{a^2-x^2}=\frac{1}{2}$ log. $(a+x)+\frac{1}{2}$ log. $(a-x)$.

(7) $a^3\times\sqrt[4]{a^3}=a^3\times a^{\frac{3}{4}}=a^{\frac{15}{4}}$; and log. $\left(a^{\frac{15}{4}}\right)=\frac{15}{4}$ log. a, or, $3\frac{3}{4}$ log. a.

(8) $$\frac{\sqrt{a^2-x^2}}{(a+x)^2}=\frac{\sqrt{a^2-x^2}}{\sqrt{(a+x)^4}}=\sqrt{\frac{(a+x)(a-x)}{(a+x)(a+x)^3}}=\sqrt{\frac{a-x}{(a+x)^3}};$$

log. $\frac{a-x}{(a+x)^3}=$ log. $(a-x)-$ log. $(a+x)^3$,
$=$ log. $(a-x)-3$ log. $(a+x)$;

hence log. $\sqrt{\frac{a-x}{(a+x)^3}}$, or log. $\frac{\sqrt{a^2-x^2}}{(a+x)^2}$
$=\frac{1}{2}\{$log. $(a-x)-3$ log. $(a+x)\}$;
or, $=\frac{1}{2}$ log. $(a-x)-\frac{3}{2}$ log $(a+x)$; but the first form is the best.

Article 370.

(1) Since $14=2\times7$ $\therefore$ log. $14=$ log. $2+$ log. 7,
Since $15=3\times5$ $\therefore$ log. $15=$ log. $3+$ log. 5;
Since $16=2^4$ $\therefore$ log. $16=$ (log. $2)\times4$;
Since $18=3^2\times2$ $\therefore$ log. $18=$ (log. $3)\times2+$ log. 2;
Since $20=2^2\times5$ $\therefore$ log. $20=$ (log. $2)\times2+$ log. 5;
Since $21=3\times7$ $\therefore$ log. $21=$ log. $3+$ log. 7;
Since $24=2^3\times3$ $\therefore$ log. $24=$ (log. $2)\times3+$ log. 3;
Since $25=5^2$ $\therefore$ log. $25=$ (log. $5)\times2$;
Since $27=3^3$ $\therefore$ log. $27=$ (log. $3)\times3$;
Since $28=2^2\times7$ $\therefore$ log. $28=$ (log. $2)\times2+$ log. 7;
Since $30=3\times10$ $\therefore$ log. $30=$ log. $3+$ log. 10.

(2) The numbers will evidently be those that can be formed by multiplying together any two or more of the factors 2, 3, 5, 7, either of which may be taken more than once if necessary, thus,

2^5, 5×7, $3^2\times2^2$, $2^3\times5$, $2\times3\times7$, $3^2\times5$, $2^4\times3$, 7^2, $5^2\times2$, $3^3\times2$, $2^3\times7$, $2^2\times3\times5$, $3^2\times7$, 2^6, $2\times5\times7$, $2^3\times3^2$, $5^2\times3$, $2^4\times5$, 3^4, $2^2\times3\times7$, $3^2\times2\times5$, $2^5\times3$, $7^2\times2$.

Article 377.

REMARK.—The pupil will find the logarithm of 2, as given in all the tables in common use to be .30103000; from this he may perhaps infer that there is some defect in the calculations in this article in the Algebra On the contrary, however, the result there given, as far as it is carried, is absolutely correct, the logarithm of 2 to 20 places of decimals being .30102999566398119521. (See Hutton's Tables.)

(1) To find the logarithm of 3,

$$\text{Log. P} = \text{log. } 2 \ldots\ldots = .30102999;$$

$$\frac{2A}{2P+1} = \frac{.86858896}{5} \ldots\ldots = .17371779; \text{ (B)}$$

$$\frac{B}{3(2P+1)^2} = \frac{.17371779}{3\times5^2} \ldots\ldots = .00231623; \text{ (C)}$$

$$\frac{3C}{5(2P+1)^2} \ldots\ldots = .00005559; \text{ (D)}$$

$\frac{5D}{7(2P+1)^2}$ . . . . $=.00000159$; (E)

$\frac{7E}{9(2P+1)^2}$ . . . . $=.00000005$; (F)

∴ Common log. of 3 . . . . . . $=.47712124.$

(2) To find the logarithm of 5,

Here P $=4$, and log. P$=2$ log. 2 $=.60205999$;

$\frac{2A}{2P+1} = \frac{.86858896}{9}$ . . . . $=.09650988$; (B)

$\frac{B}{3(2P+1)^2} = \frac{.09650988}{3\times 9^2}$ . . . . $=.00039716$; (C)

$\frac{3C}{5(2P+1)^2}$ . . . . $=.00000294$; (D)

$\frac{5D}{7(2P+1)^2}$ . . . . $=.00000003$; (E)

log. 5 . . . . $=.69897000.$

The last figure of the term E is taken to the nearest unit.

It is not necessary, however, except as an exercise, to calculate the common logarithm of 5, since $5=\frac{10}{2}$, and log. 5= log. 10- log. 2=1— log. 2

(3) To find the logarithm of 7.

Here P$=6$, and log. 6= log. 2+ log. 3 $=.77815123$;

$\frac{2A}{2P+1} = \frac{.86858896}{13}$ . . . . $=.06681453$; (B)

$\frac{B}{3(2P+1)^2} = \frac{.06681453}{3\times 13^2}$ . . . . $=.00013178$; (C)

$\frac{3C}{5(2P+1)^2}$ . . . . $=.00000047$; (D)

log. 7 . . . . $=.84509801.$

(4) To find the logarithm of 11.

Here P $=10$, and log. P . . . . $=1.00000000$;

$\frac{2A}{2P+1} = \frac{.86858896}{21}$ . . . . $= .04136138$; (B)

$\frac{B}{3(2P+1)^2}$ . . . . $= .00003126$ (C)

$\frac{3C}{5(2P+1)^2}$ . . . . $=.00000004$; (D)

log. 11 . . . . $=1.04139268$.

Article 379.

(1) First. No system of logarithms can have a negative base since the odd powers of a negative number are negative, and therefore the positive numbers corresponding to the odd powers of the base would not be represented.

Second. The base of a system of logarithms cannot be 1, for the simple reason that every power of 1 is 1.

(2) Calling A and A′ the moduli of two different systems whose logarithms are denoted by log. and log.′; if B and C are two numbers, from Art. 376, we have

$$\text{log. B} : \text{log}'.\ \text{B} :: \text{A} : \text{A}',$$

$$\text{log. C} : \text{log}'.\ \text{C} :: \text{A} : \text{A}',$$

whence $\text{log. B} : \text{log. C} :: \text{log}'.\text{B} : \text{log}'.\ \text{C}$,

or, $\frac{\text{log. C}}{\text{log. B}}=\frac{\text{log}'.\ \text{C}}{\text{log}'.\ \text{B}}$;

that is, *the logarithms of the same numbers, in two different systems, have the same ratio to each other.*

Example. The ratio of the common logarithm of 2 to that of 10, is $\frac{1.00000}{.30103}=3.321928$; and the ratio of the Naperian logarithm of 2 to that of 10, is $\frac{2.302585}{.693147}=3.321928$.

(3) Let N and N+1 be two consecutive numbers, the difference of their logarithms, taken in any system, will be log. (N+1)— log. N.

But (Art. 351), $\text{log. (N+1)} - \text{log. N} = \text{log.} \left(\frac{\text{N}+1}{\text{N}}\right)$

$= \text{log.} \left(1+\frac{1}{\text{N}}\right)$, a quantity which approaches to the logarithm of 1 (which is zero, Art. 367.) in proportion as $\frac{1}{\text{N}}$ decreases, that is, as N increases. Hence, *the difference of the logarithms of two consecutive numbers is less, as the numbers themselves are greater.*

Example. The difference of the logarithms of 9 and 10 is 1—.9542425=.0457575 ; and the difference of the logarithms of 999 and 1000, is 3—.9995655=.0004345.

EXPONENTIAL EQUATIONS.

Articles 382—383.

(2) $20^x=100$, ∴ x log. 20= log. 100,

whence $x=\frac{\text{log. }100}{\text{log. }20}=\frac{2.000000}{1.301030}=1.53724.$

(3) $100^x=250$, ∴ x log. 100= log. 250,

whence $x=\frac{\text{log. }250}{\text{log. }100}=\frac{2.397940}{2.000000}=1.19897.$

(3) Since $2^2=4$, and $3^3=27$, we easily see that x lies between 2 and 3, and that it is near the former. We also readily find that it is less than 2.2 ; then let us assume 2 and 2.2 for the two numbers.

First Supposition.		*Second Supposition.*	
$x=2$; log. 2=	.301030	$x=2.2$; log. 2.2=	.342423
x log. x . . =	.601060	x log. x . . . . =	.753330
true no. log. 5=	.698970	true no. log. 5 =	.698970
error	—.097910	error. . .	+.054360

Difference of results =.152270 ; diff. assumed nos. =.2;
As .152270 : .2 : : .05436 : .0713, correction,
2.2—.0713=2.1287.

By trial, we find that x is greater than 2.12, and less than 2.13 therefore, let 2.12 and 2.13 be two new assumed numbers.

First Supposition.		*Second Supposition.*	
$x=2.12$; log. 2.12=	.326336	$x=2.13$; log. 2.13=	.328380
x log. x . . . . . =	.691832	x log. x . . . . . =	.699449
true no.	.698970	true no.	.698970
error . . .	—.007138	error. . . .	+.000479

Diff. of results =.007617 ; diff. of assumed nos. =.01.
As .007617 : .01 : : .000479 : .000628 correction.
Hence $x=2.13—.000628=2.129372$ nearly.

(4) $x^x=42.8454$, log. 42.8454=1.631904.

Since $3^3=27$, and $4^4=256$, we see that x lies between 3 and 4, and that it is near the former. By a further trial it is soon found to be greater than 3.2, and less than 3.3; let these therefore be the two assumed numbers.

First Supposition.		*Second Supposition.*	
$x=3.2$; log. 3.2=	.505150	$x=3.3$; log. 3.3=	.518514
x log. x . . .	=1.616480	x log. x . . .	=1.711096
true no.	=1.631904	true no. . . .	=1.631904
error . . .	.015424	error . . .	+ .079192

As .094616 : .1 : : .015424 : .0163, correction,

hence $x=2.2+.0163=2.2163$ nearly.

By trial we find that $x=2.2163$, gives x log. $x=1.631809$, but that $x=2.2164$ gives x log. $x=1.631905$, hence $x=2.2164$ nearly.

(6) Log. $2=0.301030$, and $0.301030\times64=19.265920$, which is the logarithm of the number expressing the 64^{th} power of 2; and since the index is 19, the number of places of figures will be $19+1=20$. (Art. 358.)

(7) $a^{bx+d}=c$,

$(bx+d)$ log. $a=$ log. c;

or, bx log. $a=$ log. $c-d$ log. a;

whence $x=\dfrac{\text{log. } c-d.\text{ log. } a}{b.\text{ log. } a}$.

(8) $a^{mx}.b^{nx}=c$,

log. $(a^{mx}.b^{nx})=$ log. c;

but log. $(a^{mx}.b^{nx})=mx$ log. $a+nx$ log. b;

$\therefore$ mx log. $a+nx$ log. $b=$ log. c;

or, $x(m$ log. $a+n$ log. $b)=$ log. c;

whence $x=\dfrac{\text{log. } c}{m\text{ log. } a+n\text{ log. } b}$.

(9) $c^{mx}=a.b^{nx-1}$.

log. $(c^{mx})=$ log. $(a.b^{nx-1})=$ log. $a+(nx-1)$ log. b,

mx log. $c-nx$ log. $b=$ log. $a-$ log. b;

$x(m.$ log. $c-n.$ log. $b)=$ log. $a-$ log. b;

whence $x=\dfrac{\text{log. } a-\text{log. } b}{m.\text{ log. } c-n.\text{ log. } b}$.

(10) From the equation $m^{x-y}=n$, we have

$(x-y)$ log. $m=$ log. n, or x log. $m-y$ log. $m=$ log. n;

dividing by log. m; $x-y=$ log. $n\div$ log. $m=$ log. $\frac{n}{m}$;

from this, and the equation $x+y=a$, by adding and dividing by 2, we find $x=\frac{1}{2}(a+$ log. $n\div$ log. $m)$,

or, $x=\frac{1}{2}\left(a+\text{ log. }\frac{n}{m}\right)$.

By subtracting and dividing by 2, we find

$y=\frac{1}{2}(a-$ log. $n\div$ log. $m)=\frac{1}{2}\left(a-\text{ log. }\frac{n}{m}\right)$.

(11) From the equation $a^x.b^y=c$, we have log. $(a^x.b^y)=$ log. c

but log. $(a^x.b^y)=$ log. a^x+ log. $b^y=x$ log. $a+y$. log. b.

$\therefore$ x. log. $a+y$ log. $b=$ log. c;

and $my=nx$, or $y=\frac{nx}{m}$;

hence x log. $a+\frac{nx}{m}$ log. $b=$ log. c;

or, m. log. $a.x+n$. log. $b.x=m$. log. c;

whence $x=\frac{m.\text{ log. }c}{m.\text{ log. }a+n.\text{ log. }b}$;

$y=\frac{nx}{m}=\frac{n.\text{ log. }c}{m.\text{ log. }a+n.\text{ log. }b}$.

(12) First. log. $2000=$ log. $(1000\times2)=$ log. $1000+$ log. $2=$ $3+$ log. 2, and $2^x.3^z=2000$;

log. $(2^x.3^z)=$ log. $2000=3+$ log. 2;

log. $(2^x\,3^z)=$ log. 2^x+ log. $3^z=x$. log. $2+z$. log. 3;

$\therefore$ x. log. $2+z$. log. $3=3+$ log. 2;

and $3z=5x$, or $z=\frac{5x}{3}$;

hence x. log. $2+\frac{5x}{3}$. log. $3=3+$ log. 2;

or, 3 log. $2.x+5$ log. $3.x=3(3+$ log. $2)$;

whence $x=\frac{3(3+\text{ log. }2)}{3\text{ log. }2+5\text{ log. }3}$;

and $z=\frac{5x}{3}=\frac{5(3+\text{ log. }2)}{3\text{ log. }2+5\text{ log. }3}$.

(13) Let $a^x=z$, then $a^{2x}=z^2$, and the equation becomes
$z^2-2z=8$, or $z^2-2z+1=9$;
whence $z=\pm3+1=4$, or -2.
$\therefore\ a^x=4$, or -2,
hence x log. $a=$ log. 4, or log. (-2), but the last is inadmissible (Art. 369); also, $4=2^2$, and log. $4=2$ log. 2;

$\therefore\ x.$ log. $a=2$ log. 2, and $x=\frac{2 \text{ log. } 2}{\text{log. } a}$.

(14) Let $2^x=z$, then $2^{2x}=z^2$, and $z^2+z=12$.
From the equation $z^2+z=12$, we find $z=+3$, the negative value being omitted (Art. 369);
$\therefore\ 2^x=3$, and x log. $2=$ log. 3;

whence $x=\frac{\text{log. } 3}{\text{log. } 2}=\frac{.477121}{.301030}=1.58496$.

(15) $2a^{4x}+a^{2x}=a^{6x}$, divide each side by a^{2x};
$2a^{2x}+1=a^{4x}$,
or, $a^{4x}-2a^{2x}=1$, let $a^{2x}=z$, then
$z^2-2z=1$;
$z^2-2z+1=2$, and $z=\sqrt{2}+1$;
$\therefore\ a^{2x}=\sqrt{2}+1$;
$2x$ log. $a=$ log. $(\sqrt{2}+1)$, or $x\times2$ log. $a=$ log. $(\sqrt{2}+1)$,

whence $x=\frac{\text{log. } (\sqrt{2}+1)}{2 \text{ log. } a}$.

(16) Let $a^x=z$, then $z+\frac{1}{z}=b$, or $z^2-bz=-1$,

whence z or $a^x=\frac{b}{2}\pm\frac{1}{2}\sqrt{b^2-4}=\frac{1}{2}(b\pm\sqrt{b^2-4})$;

$\therefore\ x$ log. $a.=$ log. $\frac{1}{2}(b\pm\sqrt{b^2-4})$;

whence $x=\frac{\text{log. } \frac{1}{2}(b\pm\sqrt{b^2-4})}{\text{log. } a}$.

(17) Here $x^y=y^x(1)$, and $x^3=y^2(2)$.
Extracting the y root of both members of eq. (1), and the cube root of both members of eq. (2), we have

$x=y^{\frac{x}{y}}$, and $x=y^{\frac{2}{3}}$.

$\therefore\ y^{\frac{x}{y}}=y^{\frac{2}{3}}$, whence $\frac{x}{y}=\frac{2}{3}$, and $x=\frac{2}{3}y$;

$\therefore\ \frac{2}{3}y=y^{\frac{2}{3}}$; divide each member by $y^{\frac{2}{3}}$;

$\frac{2}{3}y^{\frac{1}{3}}=1$, or $y^{\frac{1}{3}}=\frac{3}{2}$;

cubing each side $y=(\frac{3}{2})^3=\frac{27}{8}=3\frac{3}{8}$;

$x=\frac{2}{3}y=\frac{2}{3}$ of $\frac{27}{8}=\frac{9}{4}=2\frac{1}{4}$.

(18) Here $(a^2-b^2)^{2(x-1)}=(a-b)^{2x}$.

Extracting the square root of both members, we have

$(a^2-b^2)^{(x-1)}=(a-b)^x$;

whence $(x-1)\log.(a^2-b^2)=x\log.(a-b)$;

but $\log.(a^2-b^2)=\log.[(a+b)(a-b)]=\log.(a+b)$
$+\log.(a-b)$.

$\therefore\ (x-1)\{\log.(a+b)+\log.(a-b)\}=x\log.(a-b)$;

or, $x\log.(a+b)+x\log.(a-b)-\log.(a+b)-\log.(a-b)$
$=x\log.(a-b)$;

omitting $x\log.(a-b)$ on each side, and transposing,

$x\log.(a+b)=\log.(a+b)+\log.(a-b)$;

whence $x=1+\frac{\log.(a-b)}{\log.(a+b)}$.

(19) $(a^4-2a^2b^2+b^4)^{x-1}=\{(a^2-b^2)^2\}^{x-1}=(a^2-b^2)^{2x-2}$

$=\frac{(a^2-b^2)^{2x}}{(a^2-b^2)^2}$;

and $(a-b)^{2x}(a+b)^{-2}=\frac{(a-b)^{2x}}{(a+b)^2}$;

$\therefore\ \frac{(a^2-b^2)^{2x}}{(a^2-b^2)^2}=\frac{(a-b)^{2x}}{(a+b)^2}$;

Extracting the square root of both members, we have

$\frac{(a^2-b^2)^x}{a^2-b^2}=\frac{(a-b)^x}{a+b}$;

but $(a^2-b^2)^x=\{(a+b)(a-b)\}^x=(a+b)^x(a-b)^x$;

$\therefore\ \frac{(a+b)^x(a-b)^x}{(a+b)(a-b)}=\frac{(a-b)^x}{a+b}$;

dividing both members by $(a-b)^x$, and multiplying by

$a+b$, we have $\frac{(a+b)^x}{a-b}=1$, or $(a+b)^x=a-b$;

whence x log. $(a+b)=$ log. $(a-b)$,

and $x=\frac{\text{log. }(a-b)}{\text{log. }(a+b)}$.

(20) Here $x^y=y^x(1)$, and $x^p=y^q(2)$.

From $(1)x^{\frac{y}{x}}=y$, and from $(2)x^{\frac{p}{q}}=y$;

$\therefore\ x^{\frac{y}{x}}=x^{\frac{p}{q}}$, and $\frac{y}{x}=\frac{p}{q}$;

or, $\frac{x^{\frac{p}{q}}}{x}=\frac{p}{q}$;

$\therefore\ x^{\frac{p}{q}-1}=\frac{p}{q}$, or $x^{\frac{p-q}{q}}=\frac{p}{q}$;

$\therefore\ x=\left(\frac{p}{q}\right)^{\frac{q}{p-q}}$;

$y=\frac{p}{q}x=\frac{p}{q}\left(\frac{p}{q}\right)^{\frac{q}{p-q}}=\left(\frac{p}{q}\right)^{\frac{q}{p-q}+1}=\left(\frac{p}{q}\right)^{\frac{p}{p-q}}$.

(21) Let $x^2-4x+5=z$, then $3^z=1200$, and z log. $3=$ log

whence $z=\frac{\text{log. }1200}{\text{log. }3}=\frac{3.079181}{.477121}=6.4536$.

$\therefore\ x^2-4x+5=6.4536$;

$x^2-4x+4=5.4536$;

$x-2=\pm 2.33$,

$x=2\pm 2.33=4.33$, or -0.33.

INTEREST AND ANNUITIES.

Articles 384—391.

(1) $1+r=1.06$ and log. 1.06. . . . . = .025306
.025306×100=t log. $(1+r)$ . . . . . =2.530600
log. P= log. 1 . . . . =0.000000
log. A= log. (339.30). . . . . =2.530600

(2) This example is similar to the preceding; if we multiply .025306 by 1000, the product is 25.306000, which is the

log. of the amount, and as the index is 25, the corresponding natural number will contain $25+1=26$ figures. (Art. 358.)

(3) See Art. 386, Cor. 3. For 5 per cent. $R=1.05$; for 6 per cent. $R=1.06$; for 7 per cent. $R=1.07$; for 8 per cent., $R=1.08$.

For 5 per cent., $t=\frac{\log. 2}{\log. 1.05}=\frac{.301030}{.021189}=14.206$ yrs.;

for 6 per cent., $t=\frac{\log. 2}{\log. 1.06}=\frac{.301030}{.025306}=11.8956$ yrs.;

for 7 per cent., $t=\frac{\log. 2}{\log. 1.07}=\frac{.301030}{.029384}=10.2447$ yrs.;

for 8 per cent., $t=\frac{\log. 2}{\log. 1.08}=\frac{.301030}{.033424}=9.0064$ yrs.

(4) See Art. 386, Cor. 3. Here $m=10$, and $R=1.05$.

$\therefore\ t=\frac{\log. 10}{\log. 1.05}=\frac{1.000000}{.021189}=47.19$ yrs.

(5) Let $x=$ the sum, then (Art. 386), $M=P.R^t$, and $P=x.R^t$:

whence $\frac{M}{P}=\frac{P.R^t}{x.R^t}=\frac{P}{x}$, and $\therefore\ x=\frac{P^2}{M}$.

(6) Let x, y, z, denote the three shares, then we shall have

$x+y+z=P$;

also, $x.R^a=y.R^b=z.R^c$, are the equations of condition;

whence $y=R^{a-b}x$, and $z=R^{a-c}x$;

$\therefore\ x+R^{a-b}x+R^{a-c}x=P$;

whence $x=\frac{P}{1+R^{a-b}+R^{a-c}}$.

Similarly, $y=\frac{P}{1+R^{b-a}+R^{b-c}}$, and $z=\frac{P}{1+R^{c-a}+R^{c-b}}$.

(7) If we take out the logarithm of 1.06, and multiply it by 20, and take out the number corresponding thereto, we shall have 3.20713546. Subtracting 1 from this, and dividing by .06, the quotient is 36.785591, which is the amount of an annuity of \$1, for 20 years, at 6 per cent. Then multiplying this by 120 the product is \$4414.27.

In finding the 20^{th} power of 1.06 the tables of logarithms in common use give a result too small. The learner may satisfy himself of this by actually involving 1.06 to the 20^{th} power. The formula for solving the question is found in Art. 390.

(8) See the Formula, Art. 391. $R^t=(1.05)^{30}=4.321942$, and

$\frac{1}{R^t}=.23137746$; $1-\frac{1}{R^t}=.76862254$;

$\frac{1}{R-1}\left(1-\frac{1}{R^t}\right)=\frac{1}{.05}(.76862254)=15.37245$, the present value of an annuity of \$1, to be paid for 30 years:

$15.37245\times250=3843.11+$.

(9) Here $p=\frac{a}{rR^n}\left(1-\frac{1}{R^t}\right)$ (Art. 392); $R=1.05$, $n=10$, and

$t=20$; $R^t=(1.04)^{20}=2.191123$;

$\frac{1}{R^t}=.45638697$; $1-\frac{1}{R^t}=.54361303$;

$R^n=(1.04)^{10}=1.480244$,

$rR^n=.05920976$, and $\frac{1}{rR^n}\left(1-\frac{1}{R^t}\right)=9.181138$;

$9.181138\times112.50=\$1032.87+$.

(10) The amount of a\$ at compound interest for n years, r being the rate per cent., is $a(1+r)^n$.

The amount of an annuity of b\$, for the same period, at the same rate, is $b\frac{(1+r)^n-1}{r}$ (Arts. 386 and 390).

$\therefore\ b\frac{(1+r)^n-1}{r}=a(1+r)^n$;

or, $b(1+r)^n-b=ra(1+r)^n$;

or, $b(1+r)^n-ra(1+r)^n=b$;

or, $(b-ra)(1+r)^n=b$;

or, $\log.(b-ra)+n\log.(1+r)=\log.b$;

$n\log.(1+r)=\log.b-\log.(b-ra)$;

$$n=\frac{\log.b-\log.(b-ra)}{\log.(1+r)}.$$

This formula also solves the following problem: "What sum must be paid annually to sink a given debt in a certain number of years, the interest on said debt being payable annually."

GENERAL THEORY OF EQUATIONS.

Article 396.

NOTE.— Although the *Synthetic Method of Division* is not explained till Article 409, page 356, yet we shall employ it, instead of the common method, on account of its conciseness. The Teacher who prefers to use the Synthetic method, can require his pupils to study Art. 409, before commencing the theory of equations.

(1) $1-11+23+35 \ \underline{|-1}$, since $x+1$ is the divisor,

$\underline{-\ 1+12-35}$

$1-12+35+0.$

Ans. $x^2-12x+35=0.$

(2) $1-9+26-24 \ \underline{|+3}$, since $x-3$ is the divisor,

$\underline{+3-18+24}$

$1-6+\ 8+0$

$\therefore\ x^2-6x+8=0$;

whence (Art. 231), $x=4$ or 2.

(3) $1\pm0-7+6 \ \underline{|+2}$, since $x-2$ is the divisor,

$\underline{+2+4-6}$

$1+2-3+0.$

$\therefore\ x^2+2x-3=0$, and (Art. 231), $x=1$, or -3.

(4) $1+2-41-42+360 \ \underline{|+3}$, since $x-3$ is a divisor,

$\underline{+3+15-78-360}$

$1+5-26-120+0 \ \underline{|-4}$, since $x+4$ is a divisor,

$\underline{-4-\ 4+120}$

$1+1-30+0.$

$\therefore\ x^2+x-30=0$, and (Art. 231), $x=5$, or -6.

(5) $1-3-5+9-2 \ \underline{|+1}$, since $x-1$ is a divisor,

$\underline{+1-2-7+2} \ \ \underline{|-2}$, since $x+2$ is a divisor,

$1-2-7+2+0$

$\underline{-2+8-2}$

$1-4+1+0.$

$\therefore\ x^2-4x+1=0$ and (Art 231), $x=2+\sqrt{3}$, or $2-\sqrt{3}$.

Article 398.

(2) $x=2 \therefore x-2=0,$

$x=3 \therefore x-3=0,$

$x=-5 \therefore x+5=0.$

$\therefore (x-2)(x-3)(x+5)=x^3-19x+30=0.$

(3) $x=\ \ 3 \therefore x-3=0,$

$x=-2 \therefore x+2=0,$

$x=\ \ 7 \therefore x-7=0.$

$\therefore (x-3)(x+2)(x-7)=x^3-8x^2+x+42=0.$

(4) $x=\ \ 0 \therefore x-0=0,$

$x=-1 \therefore x+1=0,$

$x=\ \ 2 \therefore x-2=0,$

$x=-5 \therefore x+5=0.$

$\therefore (x-0)(x+1)(x-2)(x+5)=x^4+4x^3-7x^2-10x=0$

(5) $x=-2 \therefore x+2=0,$

$x=+4 \therefore x-4=0,$

$x=+4 \therefore x-4=0.$

$\therefore (x+2)(x-4)(x-4)=x^3-6x^2+32=0$

(6) $x=1+\sqrt{3} \therefore x-1-\sqrt{3}=0,$

$x=1-\sqrt{3} \therefore x-1+\sqrt{3}=0.$

$\therefore (x-1-\sqrt{3})(x-1+\sqrt{3})=x^2-2x-2=0.$

(7) $x=1+\sqrt{2} \therefore x-1-\sqrt{2}=0,$

$x=1-\sqrt{2} \therefore x-1+\sqrt{2}=0,$

$x=2+\sqrt{3} \therefore x-2-\sqrt{3}=0,$

$x=2-\sqrt{3} \therefore x-2+\sqrt{3}=0.$

$\therefore (x-1-\sqrt{2})(x-1+\sqrt{2})(x-2-\sqrt{3})(x-2+\sqrt{3})$

$=(x^2-2x-1)(x^2-4x+1)=x^4-6x^3+8x^2+2x-1=0.$

(8) It has been shown, see Art. 398, that the coëfficient of the fourth term is equal to the sum of the products of all the roots taken three and three with their signs changed. The roots with their signs changed are $+2$, $+1$, -1, -3,

-4, and the sum of their products taken three and three is $(2\times1\times-1)+(2\times1\times-3)+(2\times1\times-4)$
$+(2\times-1\times-3)+(2\times-1\times-4)+(2\times-3\times-4)$
$+(1\times-1\times-3)+(1\times-1\times-4)+(1\times-3\times-4)$
$+(-1\times-3\times-4)=-2-6-8+6+8+24+3+4+12$
$-12=29$; and since x^5 appears in the 1^{st} term, $29x^2$ is the fourth term.

(9) It is evident there will be 7 terms in the required equation, hence, the middle term will be the 4^{th}, hence it is required to find the sum of the products of the numbers 5, 3, 1, -1, -2, -4, taken three and three with their signs changed. But the shortest method is to find the product of the several binomial factors, thus,

$(x-5)(x-3)(x-1)(x+1)(x+2)(x+4)$
$=(x^2-8x+15)(x^2+6x+8)(x^2-1)=x^6-2x^5-26x^4+28x^3$
$+145x^2-26x-120$, where the middle term is $28x^3$.

(10) We may take any two numbers as the other two roots, but the equation will be of the simplest form if we suppose the roots to be $+\sqrt{2}$, and $-\sqrt{-3}$, since this assumption will cause the middle terms of the binomial factors to cancel each other. Thus,

$(x-\sqrt{2})(x+\sqrt{2})(x+\sqrt{-3})(x-\sqrt{-3})=(x^2-2)(x^2+3)$
$=x^4+x^2-6=0$.

Article 400.

(1) Since $x^2-2x-24=0$ is the same as $x^2+2x-24=0$, except that the sign of the 2^{nd} term is changed, (the fourth being wanting,) and since the roots of the latter are $+4$, and -6, therefore (Art. 400), the roots of the former are -4 and $+6$.

(2) Since $x^3+3x^2-10x-24=0$, is the same as $x^3-3x^2-10x+24=0$, with the signs of the alternate terms changed, and since the roots of the latter are 2, -3 and 4. therefore (Art. 400), the roots of the former are -2. $+3$, -4.

Article 401.

(1) Dividing the given equation by $x+6=x-(-6)$, the quotient is $x^2-6x+10=0$, of which the roots are $3+\sqrt{-1}$ and $3-\sqrt{-1}$.

(2) Dividing the given equation by $x-(-4)=x+4$, the quotient is $x^2-4x+1=0$, of which the roots are $2+\sqrt{3}$, and $2-\sqrt{3}$

(3) Since $x=3+\sqrt{2}$, and $x=3-\sqrt{2}$, therefore $(x-3-\sqrt{2})(x-3+\sqrt{2})=x^2-6x+7$; dividing the given equation by this, the quotient is $x^2+2x-2=0$, of which the roots are $-1\pm\sqrt{3}$.

(4) Since one root is $2-\sqrt{3}$, therefore (Art. 401, Cor. 1,) $2+\sqrt{3}$ is another root,
$(x-2+\sqrt{3})(x-2-\sqrt{3})=x^2-4x+1$, and dividing the given equation by this, the quotient is $x-3$; hence $x-3=0$, and $x=+3$.

(5) Since $-\frac{1}{2}(3+\sqrt{-31})$ is one root of the given equation, therefore (Art. 401), $-\frac{1}{2}(3-\sqrt{-31})$ is another root;
$[x+\frac{1}{2}(3+\sqrt{-31})][x+\frac{1}{2}(3-\sqrt{-31})]=x^2+x(3)$ $+\frac{1}{4}[9-(-31)]=x^2+3x+10$, and dividing the given equation by this, the quotient is x^2-3x-4; hence $x^2-3x-4=0$, and $x=4$, or -1.

(6) Since $+\sqrt{2}$ is one root of the given equation, therefore (Art. 410, Cor. 2), $-\sqrt{2}$ is another root, and three of the binomial factors of the given equation are
$(x-\sqrt{2})(x+\sqrt{2})(x-3)=x^3-3x^2-2x+6$.
Dividing the given equation by this, the quotient is $x^2-7x+10$; hence $x^2-7x+10=0$; from which $x=2$, and 5.

Article 403.

Ex. 2. If we substitute 5 for x in the equation $x^3-5x^2-x+1=0$, we have $-4=0$, and if we substitute 6 for x, we have $+31=0$, and since the results have contrary signs, one root lies between 5 and 6, that is, 5 is the first figure of one of the roots

TRANSFORMATION OF EQUATIONS.

Article 405.

(1) Here $x^4+7x^2-4x+3=0$; let $x=\frac{y}{3}$, then

$\frac{y^4}{81}+\frac{7y^2}{9}-\frac{4y}{3}+3=0$, multiply by 81, to clear of fractions, $y^4+63y^2-108y+243=0$.

(2) Here $x^4+2x^3-7x-1=0$; let $x=\frac{y}{5}$, then

$\frac{y^4}{625}+\frac{2y^3}{125}-\frac{7y}{5}-1=0$, and clearing of fractions

$y^4+10y^3-875y-625=0$.

(3) Here $x^3-3x^2+4x+10=0$; let $x=2y$, then

$8y^3-12y^2+8y+10=0$, or, dividing by 2,

$4y^3-6y^2+4y+5=0$.

(4) Here $x^3+18x^2+99x+81=0$; let $x=3y$, and change the signs of the alternate terms (Art. 400), then

$27y^3-18\times9y^2+99\times3y-81=0$, or dividing by 27,

$y^3-6y^2+11y-3=0$.

(5) Here $x^3-2x^2+\frac{1}{3}x-10=0$; let $x=\frac{1}{3}y$, then

$\frac{y^3}{27}-\frac{2y^2}{9}+\frac{1}{9}y-10=0$; clearing of fractions,

$y^3-6y^2+3y-270=0$.

Articles 406—407.

(1) Here $x^3-7x+7=0$; let $y=x-1$, then $x=y+1$.

$\therefore (y+1)^3-7(y+1)+7=0$, or $y^3+3y^2-4y+1=0$.

(2) Here $x^4-3x^3-15x^2+49x-12=0$, let $y=x-3$, then $x=y+3$; $\therefore (y+3)^4-3(y+3)^3-15(y+3)^2+49(y+3)-12=0$, or, by developing and reducing,

$y^4+9y^3+12y^2-14y=0$.

(3) Here $x^3-6x^2+8x-2=0$, and $x=y+2$.

$\therefore (y+2)^3-6(y+2)^2+8(y+2)-2=0$, or, reducing,

$y^3-4y-2=0$.

(4) $x^2+2px-q=0$. Here $A=+2p$, $n=2$.
$\therefore r=-p$; hence $x=y-p$,
substituting; $(y-p)^2+2p(y-p)-q=0$, or, reducing,
$y^2-p^2-q=0$.

Articles 408—410.

(4) Let $y=x-3$, then it is required to divide the given equation, and the successive quotients, by $x-3$.

1	±0	—27	—36	(+3, since the divisor is $x-3$.
	+3	+ 9	—54	
	+3	—18	—90 ∴ —90= 1st R	
	+3	+18		
	+6	+ 0 ∴ 0= 2nd R.		
	+3			
	+9 ∴ +9= 3rd R.			

Ans. $y^3+9y^2-90=0$.

(5)

1	±0	—27	—14	+120	(+3, since the divisor is $x-3$,
	+3	+ 9	—54	—204	
	+3	—18	—68	— 84 ∴ —84= 1st R.	
	+3	+18	— 0		
	+6	+ 0	—68 ∴ —68= 2nd R.		
	+3	+27			
	+9	+27 . . +27= 3rd R.			
	+3				
	+12 ∴ +12= 4th R.				

Ans. $y^4+12y^3+27y^2-68y-84=0$.

(6)

1	—18	— 32	+ 17	+ 9	(+5, since the divisor is $x-5$,
	+ 5	— 65	— 485	—2340	
	—13	— 97	— 468	—2331 ∴ —2331= 1st R.	
	+ 5	— 40	— 685		
	— 8	—137	—1153 ∴ —1153= 2nd R.		
	+ 5	— 15			
	— 3	—152 ∴ —152= 3rd R.			
	+ 5				
	+ 2 ∴ +2= 4th R.				

Ans. $y^4+2y^3-152y^2-1153y-2331=0$.

(7) We shall first diminish the roots by 1, and then by .2, indicating the remainders after each transformation by stars.

1	—6	+7.4	+ 7.92	—17.872	— .79232 (+1.2
	+1	—5.	+ 2.4	+10.32	—7.552
	—5	+2.4	+10.32	— 7.552	—8.34432*
	+1	—4.	— 1.6	+ 8.72	+ .34432
	—4	—1.6	+ 8.72	+ 1.168*	—8*
	+1	—3.	— 4.6	+ .5536	
	—3	—4.6	+ 4.12*	+ 1.7216	
	+1	—2	— 1.352	+ .2784	
	—2	—6.6*	+ 2.768	+ 2.0000*	
	+1	— .16	— 1.376		
	—1*	—6.76	+ 1.392		
	+.2	— .12	— 1.392		
	—.8	—6.88	+ 0.000*		
	+.2	— .08			
	—.6	—6.96			
	+ 2	— .04			
	—.4	—7.*			
	+.2				
	—.2				
	+.2				
	0*				

Ans. $y^5-7y^3+2y-8=0$.

(8) Here $A=-6$, $n=3$, $\therefore r=2$; hence $x=y+2$, or $y=x-2$.

1	—6	+7	—2	(+2, since the divisor is $x-2$.
	+2	—8	—2	
	—4	—1	—4 $\therefore$ —4= 1st R.	
	+2	—4		
	—2	—5 $\therefore$ —5= 2nd R.		
	+2			
	0 $\therefore$ 0= 3rd R. *Ans.* $y^3-5y-4=0$.			

(9) Here $A=-6$, $n=3$, $\therefore r=2$; hence $x=y+2$, or $y=x-2$.

$$\begin{array}{lllll}
1 & -6 & \pm 0 & +5 & (+2, \text{ since the divisor is } x-2 \\
 & \underline{+2} & \underline{-8} & \underline{-16} & \\
 & -4 & -8 & -11 & \therefore\ -11 = 1^{st}\text{ R.} \\
 & \underline{+2} & \underline{-4} & & \\
 & -2 & -12 & \therefore\ -12 = 2^{nd}\text{ R.} & \\
 & \underline{+2} & & & \\
 & 0 & \therefore\ 0 = 3^{rd}\text{ R.} & \textit{Ans. } y^3-12y-11=0. &
\end{array}$$

(10) Here $A=-6$, $n=3$, $\therefore\ r=2$;

hence $x=y+2$, or $y=x-2$.

$$\begin{array}{lllll}
1 & -6 & +12 & +19 & (+2, \text{ since the divisor is } x-2 \\
 & \underline{+2} & \underline{-8} & \underline{+8} & \\
 & -4 & +4 & +27 & \therefore\ +27 = 1^{st}\text{ R.} \\
 & \underline{+2} & \underline{-4} & & \\
 & -2 & 0 & \therefore\ 0 = 2^{nd}\text{ R.} & \\
 & \underline{+2} & & & \\
 & 0 & \therefore\ 0 = 3^{rd}\text{ R.} & \textit{Ans. } y^3+27=0. &
\end{array}$$

(11) Dividing each term by 3, the given equation becomes

$x^3+5x^2+\frac{25}{3}x-1=0.$

Here $A=5$, $n=3$, $\therefore\ r=-\frac{5}{3}$;

hence $x=y-\frac{5}{3}$, or $y=x+\frac{5}{3}$.

$$\begin{array}{lllll}
1 & +5 & +\frac{25}{3} & -1 & (-\frac{5}{3}, \text{ since the divisor is } x+\frac{5}{3}. \\
 & \underline{-\frac{5}{3}} & \underline{-\frac{50}{9}} & \underline{-\frac{125}{27}} & \\
 & +\frac{10}{3} & +\frac{25}{9} & -\frac{152}{27} & \therefore\ -\frac{152}{27} = 1^{st}\text{ R.} \\
 & \underline{-\frac{5}{3}} & \underline{+\frac{25}{9}} & & \\
 & +\frac{5}{3} & 0 & \therefore\ 0 = 2^{nd}\text{ R.} & \\
 & \underline{-\frac{5}{3}} & & & \\
 & 0 & \therefore\ 0 = 3^{rd}\text{ R.} & \textit{Ans. } y^3-\frac{152}{27}=0, \text{ or } 27y^3-152=0 &
\end{array}$$

(12) Here $A=-6$, $B=+9$, $n=3$,

$\frac{1}{2}n(n-1)r^2+(n-1)Ar+B=\frac{3}{2}(2)r^2+(2)\times-6r+9=0,$

or, $r^2-4r+3=0$, whence $r=3$, or 1.

hence $x=y+3$, or $y+1$, and $y=x-3$, or $x-1$.

$$\begin{array}{llll}
1 & -6 & +9 & -20 \quad (+3, \text{ since the divisor is } x-3. \\
 & +3 & -9 & +0 \\
 & -3 & 0 & -20 \\
 & +3 & 0 & \\ \hline
 & 0 & 0 & \\
 & +3 & & \\ \hline
 & +3 & & \qquad \textit{Ans. } y^3+3y^2-20=0.
\end{array}$$

$$\begin{array}{lllll}
\text{or, } 1 & -6 & +9 & -20 \quad (+1, \text{ since the divisor is } x-. \\
 & +1 & -5 & +4 \\ \hline
 & -5 & +4 & -16 \\
 & +1 & -4 & \\ \hline
 & -4 & 0 & \\
 & +1 & & \\ \hline
 & -3 & & \qquad \textit{Ans. } y^3-3y^2-16=0.
\end{array}$$

(12) Here $A=-4$, $B=5$, $n=3$.

$\frac{1}{2}n(n-1)r^2+(n-1)Ar+B=3r^2-8r+5=0$, and $r=\frac{5}{3}$ or 1 hence $x=y+\frac{5}{3}$, or $y+1$, and $y=x-\frac{5}{3}$, or $x-1$.

$$\begin{array}{llll}
1 & -4 & +5 & -2 \quad (+\frac{5}{3}, \text{ since the divisor is } x-\frac{5}{3}. \\
 & +\frac{5}{3} & -\frac{35}{9} & +\frac{50}{27}, \\ \hline
 & -\frac{7}{3} & +\frac{10}{9} & -\frac{4}{27} \\
 & +\frac{5}{3} & -\frac{10}{9} & \\ \hline
 & -\frac{2}{3} & 0 & \\
 & +\frac{5}{3} & & \\ \hline
 & +1 & &
\end{array}
\qquad
\begin{array}{llll}
\text{or } 1 & -4 & +5 & -2 \quad (+1, \\
 & +1 & +3 & +2 \\ \hline
 & -3 & +2 & 0 \\
 & +1 & +2 & \\ \hline
 & -2 & 0 & \\
 & +1 & & \\ \hline
 & -1 & &
\end{array}$$

Ans. $y^3+y^2-\frac{4}{27}=0$. *Ans.* $y^3-y^2=0$.

EQUAL ROOTS.

Article 414.

(2) Here $x^3-2x^2-15x+36=0$,

$3x^2-4x-15=$ 1st derived polynomial, and the greatest common divisor of this and the given equation (Art. 108,) is $x-3$; hence $x-3=0$, and $x=+3$, therefore $+3$ and $+3$ are two roots of the given equation.

Dividing the given equation by $(x-3)(x-3)$ the quotient is $x+4$, hence $x+4=0$, and $x=-4$.

(3) Here $x^4-9x^2+4x+12=0$,

$4x^3-18x+4=$ 1st derived polynomial, and the greatest common divisor of this and the given equation is $x-2$; hence $x=+2$, and $+2$. Dividing the given equation by $(x-2)(x-2)$ the quotient is x^2+4x-3; hence $x^2+4x-3=0$ from which we find $x=-1$, and -3.

(4) Here $x^4-6x^3+12x^2-10x+3=0$,

$4x^3-18x^2+24x-10=$ 1st derived polynomial, and the greatest common divisor of this and the given equation is x^2-2x+1; but $x^2-2x+1=(x-1)^2$, therefore the given equation has three roots, each equal to 1.

Dividing the given equation by $(x-1)(x-1)(x-1)$ the quotient is $x-3$, hence $x-3=0$, and $x=3$.

The operation of dividing by $x-1$ should be performed by synthetic division on account of its brevity, thus,

1	−6	+12	−10	+3	(+1
	+1	−5	+7	−3	
	−5	+7	−3	0	
	+1	−4	+3		
	−4	+3	0		
	+1	−3			
	−3	0.	Quotient $=x-3$.		

(5) Here $x^4-7x^3+9x^2+27x-54=0$,

$4x^3-21x^2+18x+27=$ the first derived polynomial, and the greatest common divisor of this and the given equation is x^2-6x+9; but $x^2-6x+9=(x-3)^2$, therefore the equation has three roots, each equal to 3.

Dividing the given equation by $(x-3)(x-3)(x-3)$, the quotient is $x+2$, hence $x+2=0$, and $x=-2$.

(6) Here $x^4+2x^3-3x^2-4x+4=0$,

$4x^3+6x^2-6x-4=$ 1st derived polynomial, and the greatest common divisor of this and the given equation is $x^2+x-2=(x+2)(x-1)$; therefore the equation contains two factors of the form $x+2$, and of the form $x-1$, hence the four roots are -2, -2, $+1$, $+1$.

If the learner does not readily see that $x^2+x-2=(x+2)(x-1)$, et him place it equal to zero and find the roots.

(7) Here $x^4-12x^3+50x^2-84x+49=0$,

$4x^3-36x^2+100x-84=$ 1st derived polynomial, and the greatest common divisor of this and the given equation is x^2-6x+7.

Placing this equal to zero, we find its roots are $3+\sqrt{2}$, and $3-\sqrt{2}$, that is, $x^2-6x+7=(x-3-\sqrt{2})(x-3+\sqrt{2})$, hence the four binomial factors of the given equation are $(x-3-\sqrt{2})(x-3-\sqrt{2})(x-3+\sqrt{2})(x-3+\sqrt{2})$, and the four roots are $3+\sqrt{2}$, $3+\sqrt{2}$, $3-\sqrt{2}$, $3-\sqrt{2}$.

(8) Here $x^5-2x^4+3x^3-7x^2+8x-3=0$,

$5x^4-8x^3+9x^2-14x+8=$ 1st derived polynmoial, and the greatest common divisor of this and the given equation is $x^2-2x+1=(x-1)^2$, therefore the equation has three roots, each equal to $+1$.

Dividing the given equation by $(x-1)(x-1)(x-1)$ the quotient is x^2+x+3, thus,

1	−2	+3	−7	+8	−3	+1
	+1	−1	+2	−5	+3	
	−1	+2	−5	+3	0 = 1st R.	
	+1	+0	+2	−3		
	0	+2	−3	0 = 2nd R.		
	+1	+1	+3			
	+1	+3	0 = 3rd R.			

$\therefore x^2+x+3=0$, from which $x=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-11}$.

As learners sometimes experience difficulty in finding the greatest common divisor in this example, we will here give the operation.

Multiplying the given equation by 5, the operation is as follows:

$$
\begin{array}{rl}
5x^5-10x^4+15x^3-35x^2+40x-15 & \big|\, 5x^4-8x^3+9x^2-14x+8 \\
5x^5-\ 8x^4+\ 9x^3-14x^2+\ 8x & \big|\, x-2 \\
\hline
-\ 2x^4+\ 6x^3-\ 21x^2+\ 32x-15 & \\
-10x^4+30x^3-105x^2+160x-75 & \\
-10x^4+16x^3-\ 18x^2+\ 28x-16 & \\
\hline
+14x^3-\ 87x^2+132x-59 &
\end{array}
$$

Multiply the first divisor by 14.

$$70x^4-112x^3+126x^2-196x+112 \quad |\underline{14x^3-87x^2+132x-59}$$
$$\underline{70x^4-435x^3+660x^2-295x} \qquad\qquad |\underline{5x+323}$$
$$+323x^3-534x^2+\ 99x+112$$
$$\times \text{ by } 14, \quad 4522x^3-7476x^2+1386x+1568$$
$$\underline{4522x^3-28101x^2+42636x-19057}$$
$$20625x^2-41250x+20625,$$
$$\text{or, } 20625(x^2-2x+1)$$

x^2-2x+1 will be found to divide $14x^3-87x^2+132x-59$, and it is therefore the greatest common divisor required.

(9) Here $x^6+3x^5-6x^4-6x^3+9x^2+3x-4=0$.

$6x^5+15x^4-24x^3-18x^2+18x+3=$ 1st derived polynomial,

$30x^4+60x^3-72x^2-36x+18 \qquad =$ 2nd derived polynomial.

We find the greatest common divisor of the given equation and the first derived polynomial is x^3-x^2-x+1; if we put this equal to zero, it is easily seen that $x=1$, then dividing by $x-1$, the quotient is x^2-1, of which the factors are $x+1$ and $x-1$, hence $x^3-x^2-x+1=(x-1)(x-1)(x+1)=(x-1)^2(x+1)$, therefore the given equation contains $x-1$ as a factor *three* times, and $x+1$ as a factor *twice*; hence three roots of the equation are $+1$, $+1$, $+1$, and two roots -1, -1.

Dividing the given equation by $(x-1)^3(x+1)^2$, the quotient is $x+4$; hence $x+4=0$, and $x=-4$.

Otherwise thus:

After finding the greatest common divisor of the given equation, and its first derived polynomial, we may proceed to find the greatest common divisor of the 1st and 2nd derived polynomials, which is $x-1$; hence, since the 2nd derived polynomial contains $x-1$ as a factor *once*, the 1st derived polynomial must contain it as a factor *twice*, and the given equation *three* times.

Also, by dividing x^3-x^2-x+1 by $(x-1)^2$, the quotient is $x+1$, which is therefore contained *twice* as a factor in the given equation. The operation of finding the greatest common divisor of the 1st and 2nd derived polynomials is quite tedious, but it enables us to determine that $x-1$ is a factor of x^3-x^2-x+1 without solving an equation of the third degree, the method of doing which has not yet been explained.

THEOREM OF STURM.

Articles 420—427.

(3) Here $X = x^3-2x^2-x+2$, and (Art. 411)

$X_1=3x^2-4x-1$.

Muitiplying X by 3, to render the first term divisible by the first term of X_1, and proceeding according to Art. 108, we have for a remainder $-14x+16$. Canceling the factor $+2$, and changing the signs (Art. 420), we have $X_2=+7x-8$. Multiplying X_1 by 7 to render the first term divisible by the first term of X_2, and proceeding as before, the remainder is -81; hence $X^3=+81$, and the series of functions is

$$X = x^3-2x^2-x+2$$
$$X_1=3x^2-4x-1$$
$$X_2=7x-8$$
$$X_3=+81$$

	X,	X_1	X_2	X_3	
For $x=-\infty$ the signs are	$-$	$+$	$-$	$+$,	3 var. ∴ k $=3$
$x=+\infty$ the signs are	$+$	$+$	$+$	$+$.	0 var. ∴ k′$=0$.

∴ k—k′=2—1=1, the number of real roots.

By substituting the whole numbers from -2 to $+3$, we find the roots are -1, $+1$, and $+2$.

(4) Here $X = 8x^3-36x^2+46x-15$, and (Art. 411),

$X_1=24x^2-72x+46$, (or $12x^2-36x+23$).

Multiplying X by 3, and dividing by X_1, the first remainder is $-36x^2+92x-45$; multiplying this by 2, and continuing the division, the remainder is $-32x+48=16(-2x+3)$, hence $X_2=2x-3$. Dividing X_1 by X_2 the remainder is -8, hence $X_3=+8$, and the series of functions is

$$X = 8x^3-36x^2+46x-15$$
$$X_1=24x^2-72x+46$$
$$X_2= 2x-3$$
$$X_3=+8.$$

For $x=-\infty$ the signs are $-+-+$, 3 var. ∴ k $=3$,

$x=+\infty$ the signs are $++++$, 0 var. ∴ k′$=0$

∴ k—k′=3—0=3, the number of real roots.

By substituting the whole numbers, from 0 to 3, we find that one variation is lost in passing from 0 to 1, one from 1 to 2, and one from 2 to 3.

(5) Here $X = x^3-3x^2-4x+11$, and (Art. 411),

$$X_1=3x^2-6x-4.$$

Multiplying X by 3, and dividing by X_1, the remainder is $-14x+29$, hence $X_2=14x-29$. Multiplying X_1 by 14, and dividing by X_2, the first remainder is $+3x-56$, multiplying this by 14, and continuing the division, the remainder is -697, hence $X_3=+697$, and the series of functions is

$$X = x^3-3x^2-4x+11$$
$$X_1= 3x^2-6x-4$$
$$X_2=14x-29$$
$$X_3=+697.$$

For $x=-\infty$ the signs are $-+-+$, 3 var. $\therefore$ $k=3$,
$x=+\infty$ the signs are $++++$, 0 var. $\therefore$ $k'=0$.
$\therefore$ $k-k'=3-0=3$, the number of real roots.

By substituting the whole numbers from -2 to $+4$, we find that one variation is lost in passing from -2 to -1, one from $+1$, to $+2$, and one from $+3$ to $+4$.

(6) Here $X=x^3-2x-5$, and $X_1=3x^2-2$.

Multiplying X by 3, and dividing by X_1, the remainder is $-4x-15$, hence $X_2=4x+15$. Multiplying X_1 by 4, and dividing by X_2, the first remainder is $-45x-8$; multiplying this by 4, and continuing the division, the remainder is $+643$, hence $X_3=-643$, and the series of functions is

$$X = x^3-2x-5$$
$$X_1=3x^2-2$$
$$X_2=4x+15$$
$$X_3=-643.$$

For $x=-\infty$ the signs are $-+--$, 2 var. $\therefore$ $k=2$,
$x=+\infty$ the signs are $+++-$, 1 var. $\therefore$ $k'=1$.
$\therefore$ $k-k'=2-1=1$, the number of real roots.

We also find that one variation is lost in passing from 2 to 3 therefore the root lies between 2 and 3.

(7) Here $X=x^3-15x-22$, and $X_1=3x^2-15$, or x^2-5.

Dividing X by $X_1=x^2-5$, the remainder is $-10x-22=2(-5x-11)$, hence $X_2=+5x+11$. Multiplying X_1 by 5, and dividing by X_2, the first remainder is $-11x-25$; multiplying this by 5, and continuing the division, the remainder is -4, hence $X_3=+4$, and the series of functions is

$$X = x^3-15x-22$$
$$X_1= x^2-5$$
$$X_2=5x+11$$
$$X_3=+4.$$

For $x=-\infty$ the signs are $-+-+$, 3 var. ∴ $k=3$,
$x=+\infty$ the signs are $++++$, 0 var. ∴ $k'=0$.
∴ $k-k'=3-0=3$, the number of real roots.

By substituting the whole numbers from -3 to $+5$, we find that two variations are lost from -3 to -2, and one from $+4$ to $+5$; we also find that -2 is a root. For $x=-2\frac{1}{2}$ there are three variations, and for $x=-2\frac{1}{4}$ there are two variations, hence one root lies between $-2\frac{1}{4}$ and $-2\frac{1}{2}$.

(8) Here $X = x^4+x^3-x^2-2x+4$, and (Art. 411), $X_1=4x^3+3x^2-2x-2$.

Multiplying X by 4, and dividing by X_1, the first remainder is $x^3-2x^2-6x+16$, multiplying this by 4, and continuing the division, the remainder is $-11x^2-22x+66=11(-x^2-2x+6)$, hence $X_2=x^2+2x-6$. Dividing X_1 by X_2 the remainder is $+32x-32=32(x-1)$, hence $X_3=-x+1$. Dividing X_2 by X_3, the remainder is -3, hence $X_4=+3$, therefore the series of functions is

$$X = x^4+ x^3-x^2-2x+4$$
$$X_1=4x^3+3x^2-2x-2$$
$$X_2=x^2+2x-6$$
$$X_3=-x+1$$
$$X_4=+3.$$

For $x=-\infty$ the signs are $+-+++$, 2 var. ∴ $k=2$.
$x=+\infty$ the signs are $+++-+$, 2 var. ∴ $k'=2$.
∴ $k-k'=2-2=0$; hence there are no real roots.

(9) Here $X=x^4-4x^3-3x+23$, and $X_1=4x^3-12x^2-3$.

Multiplying X by 4, and dividing by X_1, the remainder is

$-12x^2-9x+89$, hence $X_2=+12x^2+9x-89$. Multiplying X_1 by 3, and dividing by X_2, the first remainder is $-45x^2+89x-9$, multiplying this by 4, and continuing the division, the remainder is $+491x-1371$, hence $X_3=-491x+1371$. Multiplying X_2 by 491, and dividing by X_3, the first remainder is $20871x-43699$, multiplying this by 491, and continuing the division, the remainder is $+7157932$, hence $X_4=-7157932$, and the series of functions is

$$X = x^4-4x^3-3x+23$$
$$X_1=4x^3-12x^2-3$$
$$X_2=12x^2+9x-89$$
$$X_3=-491x+1371$$
$$X_4=-7157932.$$

For $x=-\infty$ the signs are $+-++-$, 3 var. $\therefore$ $k=3$,
$x=+\infty$ the signs are $+++--$, 1 var. $\therefore$ $k'=1$.
$\therefore$ $k-k'=3-1=2$, the number of real roots.

By substituting the whole numbers from 1 to 4, we find that one variation is lost in passing from 2 to 3, and one from 3 to 4.

(10) Here $X=x^4-2x^3-7x^2+10x+10$, and $X_1=4x^3-6x^2-14x+10$, or $2x^3-3x^2-7x+5$.

Multiplying X by 2, and dividing by X_1, the first remainder is $-x^3-7x^2+15x+20$, multiplying this by 2, and continuing the division, the remainder is $-17x^2+23x+45$, hence $X_2=17x^2-23x-45$. Multiplying X_1 by 17, and dividing by X_2, the first remainder is $-5x^2-29x+85$; multiplying this by 17, and continuing the division, the remainder is $-608x+1220=4(-152x+305)$, hence $X_3=152x-305$. Multiplying X_2 by 152, and dividing by X_3, the first remainder is $1689x-6840$, multiplying this by 152, and continuing the division, the remainder is -524535, hence $X_4=+524535$, and the series of functions is

$$X = x^4-2x^3-7x^2+10x+10$$
$$X_1= 2x^3-3x^2-7x+5$$
$$X_2=17x^2-23x-45$$
$$X_3=152x-305$$
$$X_4=+524535.$$

For $x=-\infty$ the signs are $+-+-+$, 4 var. $\therefore$ $k=4$,
$x=+\infty$ the signs are $+++++$, 0 var. $\therefore$ $k'=0$.
$\therefore$ $k-k'=4-0=4$, the number of real roots.

We also find that one variation is lost in passing from -3 to -2, one in passing from -1 to 0, and two in passing from $+2$ to $+3$.

(11) Here $X=x^5-10x^3+6x+1$, and $X_1=5x^4-30x^2+6$.

Multiplying X by 5, and dividing by X_1, the remainder is $-20x^3+24x+5$, hence $X_2=20x^3-24x-5$. Multiplying X_1 by 4, and dividing by X_2, the remainder is $-96x^2+5x+24$, hence $X_3=96x^2-5x-24$. Multiplying X_2 by 24, and dividing by X_3, the first remainder is $25x^2-456x-120$; multiplying this by 96, and continuing the division, the remainder is $-43651x-10920$, hence $X_4=43651x+10920$. Multiplying X_3 by 43651, and dividing by X_4, the first remainder is $-1266575x-1047624$; multiplying this by 43651, and continuing the division, the remainder is -1372624203024, hence $X_5=+1372624203024$. It is not necessary, however, to obtain any thing more than the sign of the last function.

The series of functions is

$$X=x^5-10x^3+6x+1$$
$$X_1=5x^4-30x^2+6$$
$$X_2=20x^3-24x-5$$
$$X_3=96x^2-5x-24$$
$$X_4=43651x+10920$$
$$X_5=+.$$

For $x=-\infty$ the signs are $-+-+-+$, 5 var. $\therefore$ k $=5$,

" $x=+\infty$ the signs are $++++++$, 0 var. $\therefore$ k$'=0$.

$\therefore$ k$-$k$'=5-0=5$, the number of real roots.

By substituting the whole numbers from -4 to $+4$, we find that one variation is lost in passing from -4 to -3, two in passing from -1 to 0, one in passing from 0 to 1, and one in passing from 3 to 4.

RESOLUTION OF NUMERICAL EQUATIONS.

RATIONAL ROOTS.

Article 429.

(2) Here $x^3-7x^2+36=0$. $+1$ and -1 are not roots.

Limit of positive roots $=1+7=8$.

Changing the signs of the alternate terms (Art. 418), the equation becomes $x^3+7x^2\pm0x-36=0$.

$\therefore$ limit of negative roots $=-(1+\sqrt[2]{36})$, or -5.

Last term $+36$

Divisors . . .	$+6$,	$+4$,	$+3$,	$+2$,	-2,	-3,	-4
Quotients . .	$+6$,	$+9$,	$+12$,	$+18$,	-18,	-12,	-9
Add 0	$+6$,	$+9$,	$+12$,	$+18$,	-18,	-12,	-9
Quotients . .	$+1$,	*	$+4$,	$+9$,	$+9$,	$+4$,	*
Add -7 . . .	-6,		-3,	$+2$,	$+2$,	-3.	
Quotients . .	-1,		-1,	$+1$,	-1,	$+1$	
Add $+1$ . . .	0,		0,	$+2$,	0,	$+2$.	

Hence the roots are $+6$, $+3$, and -2.

(3) Here $x^3-6x^2+11x-6=0$, and $+1$ is found to be a root.

Limit of positive roots $=1+6=7$.

Limit of negative roots $=0$, since when the signs of the alternate terms are changed, all the terms are positive, therefore, this equation has no positive root, and therefore the given equation has no negative root (Art. 402).

Last term -6.

Divisors	$+6$,	$+3$,	$+2$
Quotients	-1,	-2,	-3
Add $+11$	$+10$,	$+9$,	$+8$
Quotients	*,	$+3$,	$+4$
Add -6		-3,	-2
Quotients . . .		-1,	-1
Add $+1$		0,	0.

Hence the roots are $+3$, $+2$, and 1.

(4) Here $x^3+x^2-4x-4=0$, and -1 is found to be a root.

Limit of positive roots $1+\sqrt{4}=3$.

Limit of negative roots $-(1+4)=-5$.

Last term -4.

Divisors	$+2$,	-2,	-4
Quotients	-2,	$+2$,	$+1$
Add -4	-6,	-2,	-3
Quotients	-3,	$+1$,	*
Add $+1$	-2,	$+2$	
Quotients	-1,	-1	
Add $+1$	0,	0.	

Therefore the roots are $+2$, -2, -1.

(5) Here $x^3-3x^2-46x-72=0$, and $+1$ and -1 are not roots.

Limit of positive roots 72, of negative roots $-(1+\sqrt{46})$, or -8.

Last term —72

Divisors,

+72, +36, +24, +18, +12, + 9, + 8, + 6, + 4, + 3, + 2
— 2, — 3, — 4, — 6, — 8.

Quotients,

— 1, — 2, — 3, — 4, — 6, — 8, — 9, —12, —18, —24, —36,
+36, +24, +18, +12, + 9.

Add —46,

—47, —48, —49, —50, —52, —54, —55, —58, —64, —70, —82,
—10, —22, —28, —34, —37.

Quotients,

*, *, *, *, *, 6, *, *, —16, *, —41
+ 5, *, + 7, *, *, *.

Add —3,

+ 2, + 4, — 9, —19, —44.

Quotients,

— 1, — 1, — 1, *, —22

Add +1,

0, 0, 0, —21.

∴ the roots are —2, —4, +9.

(6) Here $x^3-5x^2-18x+72=0$, and $+1$ and -1 are not roots

Limit of positive roots $1+18=19$, of negative roots —72

Last term +72.

Divisors,

+18, +12, + 9, + 8, + 6, + 4, + 3, + 2, — 2, — 3,
4, — 6, — 8, — 9, —12, —18, —24, —36, —72.

Quotients,

+ 4, + 6, + 8, + 9, +12, +18, +24, +36, —36, —24,
—18, —12, — 9, — 8, — 6, — 4, — 3, — 2, — 1.

Add —18,

—14, —12, —10, — 9, — 6, 0, + 6, +18, —54, —42,
—36, —30, —27, —26, —24, —22, —21, —20, —19.

Quotients,

*, — 1, *, *, — 1, 0, + 2, + 9, +27, +14,
+ 9, + 5, *, *, + 2, *, *, *, *..

Add — 5,

*, — 6, *, *, — 6, — 5, — 3, + 4, +22, + 9,
+ 4, 0, *, *, — 3.

	Quotients,							
*	*,	*,	*, — 1,	*, — 1,	+ 2,	—11,	— 3,	
— 1,	0,	*,	*,	*.				
	Add + 1,							
0, + 1,			0,	0,	+ 3,	—10,	— 2,	

∴ +6, +3, and —4 are the roots.

(7) Here $x^4-10x^3+35x^2-50x+24=0$, and +1, is found to be one of the roots (Art. 429, Cor. 1).

The limit of the positive roots is 24, and since when we change the signs of the alternate terms, all the terms are positive, this equation has no positive roots, (Art. 402, Cor. 1), therefore the given equation has no negative roots.

Last term +24.

Divisors	+24,	+12,	+ 8,	+ 6,	+ 4,	+ 3,	+ 2.
Quotients	+ 1,	+ 2,	+ 3,	+ 4,	+ 6,	+ 8,	+12.
Add —50	—49,	—48,	—47,	—46,	—44,	—42,	—38.
Quotients	*,	— 4,	*,	*,	—11,	—14,	—19.
Add +35		—31,			+24,	+21,	+16.
Quotients		*,			+ 6,	+ 7,	+ 8.
Add —10					— 4,	— 3,	— 2.
Quotients					— 1,	— 1,	— 1.
Add +1					0,	0,	0.

∴ +4, +3, +2, and +1, are the roots.

(8) Here $x^4+4x^3-x^2-16x-12=0$, and —1 is found to be a root.

Limit of positive roots $1+\sqrt{16}=5$; of negative roots $-(1+4)=-5$.

Last term —12.

Divisors	+ 4,	+ 3,	+ 2,	— 2,	— 3,	— 4.
Quotients	— 3,	— 4,	— 6,	+ 6,	+ 4,	+ 3.
Add —16	—19,	—20,	—22,	—10,	—12,	—13.
Quotients	*,	*,	—11,	+ 5,	+ 4,	*.
Add —1			—12,	+ 4,	+ 3.	
Quotients			— 6,	— 2,	— 1.	
Add +4			— 2,	+ 2,	+ 3.	
Quotients			— 1,	— 1,	— 1.	
Add +1			0,	0,	0.	

∴ +2, —2, —3, and —1, are the roots.

(9) Here $x^4-4x^3-19x^2+46x+120=0$, and +1 and —1 are found not to be roots.

Limit of the positive roots $1+19=20$; of the negative roots $-(1+\sqrt{46})$, or —8.

Last term +120.

Divisors + 20, +15, +12, +10, + 8, + 6, + 5, + 4, + 3,
+ 2, — 2, — 3, — 4, — 5, — 6, — 8.

Quotients, + 6, + 8, +10, +12, +15, +20, +24, +30, +40,
+ 60, —60, —40, —30, —24, —20, —15.

Add +46, + 52, +54, +56, +58, +61, +66, +70, +76, +86
+106, —14, + 6, +16, +22, +26, +31.

Quotients, *, *, *, *, *, +11, +14, +19, *,
+ 53, + 7, — 2, — 4, *, *, *.

Add —19 . . . . — 8, — 5, 0,
+ 34, —12, —21, —23.

Quotients. . . . . *, — 1, 0,
+ 17, + 6, + 7, *.

Add —4 . . . . — 5, — 4,
— 13, + 2, + 3.

Quotients . . . . — 1, — 1,
*, — 1, — 1.

Add +1 . . . . 0, 0,
0, 0.

∴ +5, +4, —2, and —3 are the roots.

(10) Here $x^4+0x^3-27x^2+14x+120=0$, and +1 and —1 are not roots.

Limit of positive roots $1+\sqrt{27}$ or 7; of negative roots $-(1+27)$ or 28.

Last term +120.

Divisors, + 6, + 5, + 4, + 3, + 2, — 2, — 3, — 4, — 5,
— 6, — 8, —10, —12, —15, —20, —24.

Quotients, +20, +24, +30, +40, +60, —60, —40, —30, —24,
—20, —15, —12, —10, — 8, — 6, — 5.

Add +14, +34, +38, +44, +54, +74, —46, —26, —16, —10,
— 6, — 1, + 2, + 4, + 6, + 8, + 9.

Quotients, *, *, +11, +18, +37, +23, *, + 4, + 2,
+ 1, *, *, *, *, *, *.
Add —27 . . . . . —16, — 9, +10, — 4, —23, —25,
—26.
Quotients . . . . . . — 4, — 3, + 5, + 2, *, + 5,
*.
Add 0 . . . . — 4, — 3, + 5, + 2, + 5.
Quotients . . . . — 1, — 1, *, — 1, — 1.
Add +1 . . . . . . 0, 0, 0, 0.

∴ the roots are +4, +3, —2, and —5.

(11) Here $x^4+x^3-29x^2-9x+180=0$, and +1, and —1 are not roots.

Limit of positive roots, $1+\sqrt{29}$, or 7 ; of negative roots, $-(1+29)=-30$.

Last term +180.

Divisors, + 6, + 5, + 4, + 3, + 2, — 2, — 3, — 4, — 5
— 6, — 9, —10, —12, —15, —18, —20, —30.
Quotients, +30, +36, +45, +60, +90, —90, —60, —45, —36,
—30, —20, —18, —15, —12, —10, — 9, — 6.
Add —9, +21, +27, +36, +51, +81, —99, —69, —54, —45,
—39, —29, —27, —24, —21, —19, —18, —15.
Quotients, *, *, + 9, +17, *, *, +23, *, + 9,
*, *, *. + 2, *, *, *, *.
Add —29 . . . . —20, —12, — 6, —20.
—27,
Quotients . . . . . . — 5, — 4, + 2, + 4.
*,
Add +1 . . . . . + 4, — 3, + 3, + 5.
Quotients . . . . — 1, — 1, — 1, — 1.
Add +1 . . . . . 0, 0, 0, 0.

∴ the roots are +4, +3, —3, and —5.

(12) Here $x^3-2x^2-4x+8=0$, and +1 and —1 are not roots.

Limit of positive roots, $1+\sqrt{4}=3$; of negative roots $-(1+4)=-5$.

Last term +8.

Divisors. . . . . . . +2, —2, —4.
Quotients . . . . +4, —4, —2.

Add —4.	0, —8, —8.
Quotients	0, +4. *
Add —2	—2, +2.
Quotients	—1, —1.
Add +1	0, 0.

∴ +2 and —2 are roots, and by dividing the given equation by $(x-2)(x+2)$, the quotient is $x-2$, hence $x-2=0$, and $x=+2$, ∴ the equation has two equal roots, each of which is +2.

(13) Here $x^3+3x^2-8x+10=0$, and +1, and —1 are not roots.

Limit of positive roots, $1+\sqrt{8}$, or 4; of negative roots $-(1+8)=-9$.

Last term +10.

Divisors.	+2,	— 2,	— 5.
Quotients	+5,	— 5,	— 2.
Add —8.	—3,	—13,	—10.
Quotients	*,	*,	+ 2.
Add +3.			+ 5.
Quotient			— 1.
Add +1.			0.

∴ —5 is a root, and dividing the given equation by $x-(-5)=x+5$ the quotient is x^2-2x+2, hence $x^2-2x+2=0$, and $x=1\pm\sqrt{-1}$.

(14) Here $x^4-9x^3+17x^2+27x-60=0$, and +1, and —1 are not roots.

Limit of positive roots, $1+9=10$; of negative roots, $-(1+\sqrt[3]{27})$, or —4.

Last term —60.

Divisors,	+10,	+ 5,	+ 4,	+ 3,	+ 2,	— 2,	— 3,	— 4.
Quotients,	— 6,	—12,	—15,	—20,	—30,	+30,	+20,	+15.
Add +27,	+21,	+15,	+12,	+ 7,	— 3,	+57,	+47,	+42.
Quotients,	*,	+ 3,	+ 3,	*,	*,	*,	*,	*.
Add +17 . . .		+20,	+20.					
Quotients . . .		+ 4,	+ 5.					
Add —9 . . .		— 5,	— 4.					
Quotients .		— 1,	— 1.					
Add +1 . . .		0,	0.					

$\therefore$ $+5$, and $+4$ are roots, and by dividing the given equation by $(x-5)(x-4)$, the quotient is x^2-3, hence $x^2=3$, and $x=\pm\sqrt{3}$, $\therefore$ the four roots are $+5$, $+4$, $+\sqrt{3}$ $-\sqrt{3}$.

(15) Here $2x^3-3x^2+2x-3=0$.

Let $x=\frac{y}{2}$, then the transformed equation (Art. 405, Cor.) is $y^3-3y^2+4y-12=0$, and $+1$, and -1 are not roots.

Limit of positive roots, $1+3=4$, and since when the signs of the alternate terms are changed (Art. 400) all the terms are positive, therefore the given equation has no negative roots.

Last term	—12.
Divisors	+4, +3, +2.
Quotients	—3, —4, —6.
Add +4	+1, 0, —2.
Quotients	*, 0, —1.
Add —3	—3, —4.
Quotients	—1, —2.
Add +1	0, —1.

$\therefore$ $+3$ is a root of the transformed equation, and dividing by $y-3$ the quotient is y^2+4, hence $y^2+4=0$, and $y=\pm\sqrt{-4}=\pm 2\sqrt{-1}$.

$\therefore$ $y=+3, +2\sqrt{-1}, -2\sqrt{-1}$;

$x=\frac{y}{2}=+\frac{3}{2}, +\sqrt{-1}, -\sqrt{-1}$

(16) Here $3x^3-2x^2-6x+4=0$.

Let $x=\frac{y}{3}$, then the transformed equation (Art. 405, Cor.) is $y^3-2y^2-18y+36=0$, and $+1$, and -1 are not roots.

Limit of positive roots, $1+18=19$; of negative roots, $-(1+\sqrt{18})$, or -6.

Last term $+36$.

Divisors,										
+18,	+12,	+ 9,	+ 6,	+ 4,	+ 3,	+ 2,	— 2	— 3,	— 4,	— 6
Quotients,										
+ 2,	+ 3,	+ 4,	+ 6,	+ 9,	+12,	+18,	—18,	—12,	— 9,	— 6

Add —18,											
	—16,	—15,	—14,	—12,	— 9,	— 6,	0,	—36,	—30,	—27,	—24.
Quotients,	*,	*,	*,	— 2,	*,	— 2,	0,	+18,	+10,	*,	+ 4.
Add —2				— 4,		— 4,	— 2,	+16,	+ 8,		+ 2.
Quotients . . .				*,		*,	— 1,	— 8,	*,		*.
Add +1.							0,	— 7.			

$\therefore$ $+2$ is a root of the transformed equation, and dividing by $y-2$ the quotient is y^2-18, hence $y^2-18=0$, and $y=\pm 3\sqrt{2}$.

$\therefore$ $y=2,\ +3\sqrt{2},\ -3\sqrt{2}$.

$x=\frac{y}{3}=\frac{2}{3},\ +\sqrt{2},\ -\sqrt{2}$.

(17) Here $8x^3-26x^2+11x+10=0$.

Let $x=\frac{y}{8}$, then the transformed equation is

$y^3-26y^2+88y+640=0$, and $+1$ and -1 are not roots.

Limit of positive roots $1+26=27$; of negative roots $-(1+\sqrt[3]{640})$, or -10.

Last term $+640$.

Divisors,	+20,	+16,	+10.	+8,	+5,	+2,	—2,	—5,	—8,	—10.
Quotients,	+32,	+40,	+64,	+80,	+128,	+320,	—320,	—128,	—80,	—64.
Add +88,	+120,	+128,	+152,	+168,	+216,	+408,	—232,	—40,	+8,	+24.
Quotients,	+6,	+8,	*,	+21,	*,	+204,	—116,	+8,	—1,	*.
Add —26.	—20,	—18,		—5,		+178,	—142,	—18,	—27,	*.
Quotients,	—1,	*,		*,		+89,	+71,	*,	*.	
Add +1,	0,					+90,	+72.			

$\therefore$ $+20$ is a root of the transformed equation, and by dividing by $x-20$, the quotient is $y^2-6y-32$, hence $y^2-6y-32=0$, and $y=3\pm\sqrt{41}$.

$\therefore\ y=+20$, and $3\pm\sqrt{41}$;

$x=\frac{y}{8}=+\frac{5}{2}$, or $\frac{1}{8}(3\pm\sqrt{41})$.

(18) Here $6x^4-25x^3+26x^2+4x-8=0$.

Let $x=\frac{y}{6}$, then the transformed equation is

$y^4-25y^3+156y^2+144y-1728=0$, and $+1$ and -1 are not roots.

Limit of positive roots $1+25=26$; of negative roots $-(1+\sqrt[3]{144})$, or -7.

Last term —1728.

Divisors,	+24,	+18,	+16,	+12,	+9,	+8,	+6,
	+4,	+3,	+2,	—2,	—3,	—4,	—6.
Quotients,	—72,	—96,	—108,	—144,	—192,	—216,	—288,
	—432,	—576,	—864,	+864,	+576,	+432,	+288.
Add +144,	+72,	+48,	+36,	0,	—48,	—72,	—144,
	—288,	—432,	—720,	+1008,	+720,	+576,	+432.
Quotients,	+3,	+2,	*,	0,	*,	—9,	—24,
	—72,	—144,	—360,	—504,	—240,	—144,	—72.
Add +156,	+159,	+158,		+156,		+147,	+132,
	+84,	+12,	—204,	—348,	—84,	+12,	+84.
Quotients,	*,	*,		+13,		*,	+22,
	+21,	+4,	—102,	+174,	+28,	—3,	—14.
Add —25.				—12,			—3,
	—4,	—21,	—127,	+149,	+3,	—28,	—39.
Quotients				—1,			*,
	—1,	—7,	*,	*,	—1,	+7	*
Add +1.				0,			
	0,	—6,			0,	+8.	

$\therefore\ y=+12$, $+4$, and -3; and by dividing by $(y-12)(y-4)(y+3)$, the quotient is $y-12$, hence $y-12=0$, and $y=12$.

$\therefore\ x=\frac{y}{6}=+2$, $+2$, $+\frac{2}{3}$, and $-\frac{1}{2}$.

(19) Here $x^4-9x^3+\frac{45}{4}x^2+\frac{27}{2}x-\frac{81}{4}=0$.

Let $x=\frac{y}{2}$, then the transformed equation is

$y^4-18y^3+45y^2+108y-324=0.$

Limit of positive roots, $1+18=19$; of negative roots, $-(1+\sqrt[3]{108})$, or -6.

Last term -324.

Divisors,	+18,	+12,	+9,	+6,	+4,	+3,	+2,
	−2,	−3,	−4,	−6.			
Quotients,	−18,	−27,	−36,	−54,	−81,	−108,	−162,
	+162,	+108,	+81,	+54.			
Add +108,	+90,	+81,	+72,	+54,	+27,	0,	−54,
	+270,	+216,	+189,	+162.			
Quotients,	*,	*,	+8,	+9,	*,	0,	−27,
	−135,	−72,	*,	*.			
Add +45	. . . .	. . . .	+53,	+54,		+45,	+18,
	−90,	−27.					
Quotients,	. . . .	. . . .	*,	+9,		+15,	+9,
	+45,	+9.					
Add −18.	. . . .	. . . .	. . . .	−9,		−3,	−9
	+27,	−9.					
Quotients	. . . .	. . . .	. . . .	*,		−1,	*,
	*,	+3.					
Add +1.	. . . .	+4,				0.	

∴ +3 is a root of the transformed equation.

The first derived polynomial of the transformed equation is $4y^3-54y^2+90y+108$; now we shall find that $y-3$ is a divisor of this as well as the transformed equation, therefore +3 and +3 are two roots of the transformed equation (Art. 414), and if we divide it by $(y-3)(y-3)$ the quotient is $y^2-12y-36$, hence $y^2-12y-36=0$, and $y=6\pm6\sqrt{2}$.

$$\therefore\ y=+3,\ +3,\ +6+6\sqrt{2},\ +6-6\sqrt{2}.$$

$$x=\frac{y}{2}=+\tfrac{3}{2},\ +\tfrac{3}{2},\ +3+3\sqrt{2},\ +3-3\sqrt{2}.$$

NOTE.—This example may be solved by Art. 414, but the above is the shortest method.

HORNER'S METHOD OF APPROXIMATION.

Articles 430—434.

(1) $x^2+5x \quad -12.24=0$

1	$+5$	-12.24	$(1.8=x.$ (r = 8)
	$+1$	$+6$	
	$+6$	-6.24	
	$+1$	$+6.24$	
1	$+7^*$		
	$.8$		
	$+7.8$		

$r=\frac{V'}{T'}=\frac{6.24}{7}=.8+.$

It is readily found that x is greater than 1, and less than 2, hence 1 is the integral part of the root.

(2) $x^2+12x \qquad -35.4025=0.$

1	$+12$	-35.4025	$\|2.45=x.$ (r = 4, s = 5)
	$+2$	$+28$	
	$+14$	-7.4025^*	
	$+2$	$+6.56$	
1	$+16^*$	$-.8425^*$	
	$.4$	$+.8425$	
	$+16.4$	$.0$	
	$.4$		
1	$+16.8^*$		$r=\frac{7.4}{16}=4+.$
	05		
	$+16.85$		$s=\frac{.84}{16.8}=.05.$

When the operation gives a remainder zero, we know from Art. 395, Cor., that the exact root is obtained.

NOTE — In the solution of the succeeding problems, we shall merely present the operation, without exhibiting the work by which the successive figures of the root are obtained, since they can generally be determined mentally, as in Long Division.

(3) $4x^2-28x \qquad -61.25=0.$

4	-28	-61.25	$\|8.75=x.$
	$+32$	$+32$	
	$+4$	-29.25^*	

	32	+27.16
4	+36*	− 2.09
	2.8	+ 2.09
	+38.8	.0
	2.8	
4	+41.6*	
	.2=.05×4	
	+41.8	

(4) $8x^2-120x+394.875=0.$

8	−120	+394.875	10.125=x.
	+ 80	−400	
	− 40	− 5.125*	
	+ 80	+ 4.08	
8	+ 40*	− 1.045	
	.8	+ .8352	
	+40.8	− .2098	
	.8	+ .2098	
8	+41.6*	.0	
	.16		
	+41.76		
	.16		
8	+41.92*		
	.04=.005×8		
	+41.96		

(5) $5x^2-7.4x-16.08=0.$

5	− 7.4	−16.08	2.68=x.
	+10	+ 5.2	
	+ 2.6	−10.88	
	+10	+ 9.36	
5	+12.6*	− 1.52	
	3	+ 1.52	
	+15.6	.0	
	3		
5	+18.6*		
	.4=.08×5		
	19.0.		

(6)		x^2+x	$-1=0.$
	1	$+1$	-1 | .618034.
		.6	.96
		$+1.6$	$-$ .04
		.6	$+$ .0221
	1	$+2.2$*	.0179
		.01	.017824
		$+2.21$	.000076
		.01	
	1	2.22*	$\frac{.000076}{2.236}=.000034$ **nearly.**
		.008	
		2.228	
		.008	
	1	2.236*.	

(7)		x^2-6x	$+6=0.$
	1	-6	$+6$ | $4.73205=x.$
		$+4$	-8
		-2	-2
		$+4$	$+1.89$
	1	$+2$*	$-$ .11
		.7	$+$ .1029
		$+2.7$	$-$ .0071
		.7	
	1	$+3.4$*	$\frac{.0071}{3.46}=.00205.$
		.03	
		$+3.43$	
		.03	
	1	3.46*.	

(8)		x^3+4x^2	$-9x$	$-57.623625=0.$	
	1	$+4$	-9	-57.623625	| $3.45=x.$
		$+3$	$+21$	$+36$	
		$+7$	$+12$	-21.623625*	
		$+3$	$+30$	18.944	
		$+10$	$+42$*	$-$ 2.679625	Operation
		3	5.36	$+$ 2.679625	continued on
	1	$+13$*	$+47.36$	.0	page 275.

Operation continued on page 275.

	.4	· 5.52
	+13.4	+52.88*
	4	.7125
	+13.8	+53.5925
	4	
1	+14.2*	
	.05	
	+14.25.	

(9) $2x^3-50x+32.994306=0.$

2	±0	−50	+32.994306	4.63=x.
	+8	32	−72	
	+8	−18	−39.005694*	
	+8	+64	+36.672	
	+16	+46*	− 2.333694*	
	8	+15.12	+ 2.333694	
2	+24*	+61.12	.0	
	+ 1.2	15.84		
	+25.2	+76.96*		
	1.2	.8298		
	+26.4	+77.7898		
	1.2			
2	+27.6*			
	.06			
	+27.66.			

(10) $x^3+x^2+x-1=0.$

1	+1	+1	−1	.543689
	.5	.75	.875	
	+1.5	+1.75	− .125*	
	.5	1	.114064	
	2.0	+2.75*	.010936*	
	5	.1016	8888007	
1	2.5*	+2.8516	.002047993*	
	.04	.1032		
	2.54	+2 9548*		
	.04	.007869		
	2.58	+2.962669		

Operation continued on page 276.

	.04	.007878
1	2.62*	+2.970547*
	.003	$\frac{.002047993}{2.97}=.000689.$
	2.623	
	.003	
	2.626*.	

(11) $x^3+4x^2-5x-20=0.$

1	+4	−5	−20	2.23608.
	+2	+12	+14	
	+6	+7	−6*	
	+2	+16	+5.008	
	+8	+23*	− .992	
	2	2.04	+ .823167	
1	+10*	+25.04	− .168833	
	.2	2.08		
	+10.2	+27.12*		
	.2	.3189		
	+10.4	+27.4389		
	.2	.3198		
1	+10.6*	27.7587*		
	.03			
	+10.63			
	.03	$\frac{.168833}{27.76}=.00608.$		
	+10.66			

NOTE.— In the solution of the remaining examples in this article we shall use the abridged method employed in example 3, page 382, of the Algebra. The learner, however, who chooses to carry out the decimals fully will find no difficulty in so doing

(12) $x^3-2x-5=0.$

1	0	−2	−5	2.0945515=x.
	+2	4	+4	
	2	+2	−1*	
	2	8	+0.949329	
	4	+10*	− .050671*	
	2	.5481	+ .04451752	See page
1	+6*	+10.5481	− .00615348*	277.

	.09	.5562	557875
	6.09	+11.1043*	57473*
	.09	.02508	55800
	6.18	11.12938	1673*
	.09	.02508	1116
1	6.27*	+11.15446*	557*
		.0031	558
		11.1575	
		.003	
		+11.16*	

In finding the result true to seven places, it is not necessary to add any more figures to 6.27, as it will not alter the result.

This example is found in the work of M. Fourier, "*Analyse der Equations*," where the result is given to thirty-two places of decimals: the result to forty places, as given by Gregory is, $x=2.0945514815423265914823865405793029638576$. We apprehend few students will undertake to verify this result. The other two values of x are imaginary.

(38) $x^3+10x^2-24x-240=0$.

1	+10	−24	−240	4.8989795=x.
	+4	+56	+128	
	+14	+32	−112*	
	+4	+72	+97.792	
	+18	+104*	−14.208*	
	+4	18.24	+12.899169	
1	+22*	122.24	−1.308831*	
	.8	18.88	1.165869792	
	22.8	+141.12*	−.142961208	
	.8	2.2041	.131358087	
	23.6	143.3241	11603121	
	.8	2.2122	10218295	
1	+24.4*	+145.5363*	1384826	
	.09	.197424	1313781	
	24.49	145.733724	71045	
	.09	.197488	72938	
	24.58	+145.931212*		

[Over.]

	.09	.02222
1	+24.67*	145.95343
	.008	.02222
	24.678	+145.97565*
	.008	
	24.686	
	.008	
1	+24.694*	

(14) $x^3+12x^2-18x=216$.

1	+12	−18	−216 (4.2426407=x
	4	+64	184
	16	+46	−32*
	4	80	26.168
	20	+126*	−5.832
	4	4.84	5.468224
1	+24*	130.84	.363776
	.2	4.88	.275484488
	24.2	+135.72*	88291512
	.2	.9856	82683912
	24.4	136.7056	5607600
	.2	.9872	5512892
1	+24.6*	+137.6928*	94708
	.04	.049444	96475
	24.64	137.742244	
	.04	.049448	
	24.68	137.791692*	
	.04	.01483	
1	+24.72*	137.80652	
	.002	.0148	
	24.722	137.8213*	
	.002	.0010	
	24.724	137.8223	
	.002		
1	+24.726*		

(15) $x^4-8x^3+20x^2-15x+.5=0.$

1	—8	+20	—15	+ .5	1.284724=x
	+1	— 7	+13	—2.0	
	—7	+13	— 2	—1.5*	
	+1	— 6	+ 7	+1.0496	
	—6	+ 7	+ 5*	— .4504*	
	+1	— 5	.248	.4255385	
	—5	+ 2*	+5.248	248615*	
	+1	— .76	.104	210536	
1	—4*	1.24	5.352*	38079*	
	.2	— .72	— .032768	36820	
	—3.8	.52	5.319232	1259*	
	.2	— .68	— .0522	1052	
	—3.6	— .16*	+ 5.2670*	207	
	.2	— .2496	— .0036	210	
	—3.4	— .4096	5.2634		
	.2	— .2432	— .0036		
1	—3.2*	— .6528	+ 5.2598*		
	.08	— .24	.0006		
	—3.12	— .89*	5.26		
	.08	— .01			
	—3.04	— .90			
	.08	— .01			
	—2.96	— .91			
	.08				
1	—2.88*				

(16) $x^4+x^2-8x-15=0.$

1	+0	+ 1	— 8	—15	2.30277563=x
	+2	+ 4	+10	+ 4	
	+2	+ 5	+ 2	—11*	
	2	8	26	10.8741	
	+4	13	+28*	— .1259*	
	2	12	8.247	.09066704	
	+6	+25*	36.247	3523296	
	2	2.49	9.021	3179543	
1	+8*	+27.49	+45.268*	343753	[Over

.3	2.58	.06552	318129
8.3	30.07	45.33352	25624
.3	2.67	.06556	22724
8.6	32.74*	45.39908*	2900
.3	.02	02296	2726
8.9	32.76	45.42204	174
.3	.02	.02296	136
+9.2*	32.78	45.4450*	
	.02	23	
	32.80*	45.447	
		2	
		45.449*	

(17) $x^4-59x^2+840=0$.

NOTE.— This is a trinomial equation and may be solved as a quadratic (Art. 242), but it is placed here to be solved by Horner's method.

1	0	—59	0	+840 (4.8989795
	4	16	—172	—688
	+ 4	—43	—172	+152*
	4	32	— 44	—140.5184
	8	—11	—216*	+ 11.4816
	4	48	40.352	— 10.50697359
	12	+37*	—175.648	+ .97462641
	4	13.44	51.616	— .868978336
1	+16*	50.44	—124 032*	.105648074
	0.8	14.08	7.287849	97080804
	16.8	64.52	—116.744151	8567270
	.8	14.72	7.444827	7544964
	17.6	+79.24*	—109.299324*	1022306
	.8	1.7361	.677032	970011
	18.4	80.9761	—108.622292	52295
	.8	1.7442	.678280	53885
1	+19.2*	82.7203	—107.944012*	
	.09	1.7523	07645	
	19.29	+84.4726*	—107 86756	[See page 281.]

	.09	.1564	.07645
	19.38	84.629	—107.79111*
	.09	.156	.0059
	19.47	84.785	—107.7852
	.09	.156	.0059
1	+19.56*	+84.941*	—107.7793.

(18) $2x^4+5x^3+4x^2+3x=8002$.

2	+5	+4	+3	—8002 (7.33555
	14	133	959	+6734 (40314
	19	137	962	—1268*
	14	231	2576	1125.7932
	33	368	3538*	— 142.2068*
	14	329	214.644	119.86597542
	47	697*	3752.644	—22.34082458*
	14	18.48	220.242	20.11015620
2	+61*	715.48	3972.886*	— 2.23066838*
	.6	18.66	22.646514	2.01310385
	61.6	734.14	3995.532514	— .21756453
	6	18.84	22.703682	.20133125
	62.2	752.98*	4018.236196*	402666) 1623328(
	6	1.9038	3.79505	1610665
	62.8	754.8838	4022.03124	12663
	6	1.9056	3.79665	12080
2	+63.4*	756.7894	4025.82789*	583
	.06	1.9074	.3798	403
	63.46	758.6968*	4026.2077	180
	.06	.32	.3798	161
	63.52	759.01	4026.5875*	19
	.95	.32	38	
	63.58	759.33	4026.625	
	.06	.32	38	
2	+63.94*	759.65*	4026.663*	$x=7.3355540314$.

REMARK.— It is sometimes convenient by drawing lines, as in the preceding solution, to render separate and distinct the operation for finding each successive figure.

(19) $x^5+4x^4-3x^3+10x^2-2x=962$.

1	+4	— 3	+10	— 2	—962	3.385777
	3	+21	54	+192	+570	
	+7	+18	+64	+190	—392*	[Over

	3	30	144	624	290.21133
	10	48	208	+ 814*	—101.78867
	3	39	261	153.3711	+ 94.64260
	13	87	+469*	967.3711	—7.14607
	3	48	42.237	166.5714	6.18160
	16	+135*	511.237	1133.9425*	— .96447
	3	5.79	44.001	49.090	86793
1	+19*	140.79	555.238	1183.0325	—9654
	.3	5.88	45.792	50.096	8682
	19.3	146.67	601.030*	1233.128*	—972
	.3	5.97	12.6	3.19	868
	19.6	152.64	613.6	1236.32	—104
	.3	6.06	12.6	3.19	
	19.9	158.70*	626.2	1239.51*	
	.3		12.6	.4	
	20.2		638.8*	1239.9	
	.3			.4	$x=3.385777$
1	+20.5*			1240.3*	

(20) $x^5+2x^4+3x^3+4x^2+5x=54321.$

1	+ 2	+ 3	+ 4	+ 5	—54321 (8 414
	8	80	664	5344	42792 (455
	10	83	668	5349	—11529*
	8	144	1816	19872	11088.97344
	18	227	2484	25221*	—440.02656
	8	208	3480	2501.4336	304.1105122
	26	435	5964*	27722.4336	—135.9160478
	8	272	289.584	2620.0064	122.0290372
	34	+707*	6253.584	30342.4400*	— 13.8870106
	8	16.96	296.432	68.61122	12.2150180
1	+42*	723.96	6550.016	30411.05122	— 1.6719926
	0.4	17.12	303·344	68.68888	1.5270320
	42.4	741.08	6853.360*	30479.74010*	— .1449606
	.4	17.28	7.762	27.5192	.1527032
	42.8	758.36	6861.122	30507.2593	
	.4	17.44	7.766	27.5316	
	43.2	+775.80*	6868.888	30534.7909*	

	.4	.4	7.770	2.754	
	43.6	776.2	6876.658*	30537.545	
	.4	.4	3.1	2.754	
1	+44.0*	776.6	6879.8	20540.299*	$x=8.414455.$
		.4	3.1	.34	
		777.0	6882.9	30540.64	
		.4	3.1		
		777.4*	6886.0*.		

ADDITIONAL EXAMPLES.

NOTE.— As some instructors may desire additional examples to exercise their pupils, we subjoin the following, selected from a collection by Olinthus Gregory, Professor of Mathematics in the Royal Military Academy, Woolwich. They may also be employed as exercises in Sturm's Theorem, in finding the number and situation of the real roots.

(1) $x^3-12x=15.$ (See Art. 434–6). $\quad x=\begin{cases} 3.971960, \\ -1.576534, \\ -2.395426. \end{cases}$

(2) $x^3-13x^2+49x-45=0.$ $\quad x=\begin{cases} 5, \\ 6.64575, \\ 1.35425. \end{cases}$

(3) $x^3-6x=2.$ $\quad x=\begin{cases} 2.6016791318, \\ -2.2618022452, \\ -\ .3398768866. \end{cases}$

(4) $x^3-27x=36.$ $\quad x=\begin{cases} 5.7657415977, \\ -4.3206356862, \\ -1.4451059115. \end{cases}$

(5) $x^3-13x^2+38x+16=0.$ $\quad x=\begin{cases} 8, \\ 5.3722813, \\ -\ .3722813. \end{cases}$

(6) $x^3-7x+7=0.$

$$x=\begin{cases} 1.6920214716300958696278148\,9, \\ 1.3568958678922094438943995\,1, \\ -3.0489173395223053135222144\,0. \end{cases}$$

(7) $x^3-7035x^2+15262754x-10000730880=0.$

$x=1234$, or 2345, or 3456.

(8) $x^4-6x^2-16x+21=0$. (See Art. 429). $x=\begin{cases}3, \text{ or } 1, \text{ or} \\ -2\pm\sqrt{-3}.\end{cases}$

(9) $x^4-19x^3+123x^2-302x+200=0$. $x=\begin{cases}1.02803, \\ 4.00000, \\ 6.57653, \\ 7.39542.\end{cases}$

(10) $x^4-4x^3-3x^2-4x+1=0$.
The two real roots are $x=4.7912$ and $x=.2087$.

(11) $x^4-36x^2+72x-36=0$. $x=\begin{cases}0.872983, \\ 1.267949, \\ 4.732050, \\ -6.872983.\end{cases}$

(12) $x^4+x^3-24x^2+43x=21$. (See Art. 429). $x=\begin{cases}1, \text{ or } 3. \\ 1.1400549, \\ -6.1400549.\end{cases}$

(13) $x^4-27x^3+162x^2+356x=1200$ $x=\begin{cases}2.05607, \\ -3.00000, \\ 13.15306, \\ 14.79085.\end{cases}$

(14) $x^4-17x^2+20x-6=0$. $x=\begin{cases}4.6457513, \\ .6457513, \\ 2\pm\sqrt{2}.\end{cases}$

(15) $x^4-112.3x^3+1243.53x^2-2244.341x+1112.111=0$.
$x=1$, or 1.1, or 10.1, or 100.1.

Article 435.

TO EXTRACT THE ROOTS OF NUMBERS BY HORNER'S METHOD.

(2) To find the cube root of 34012224.

1	0	0	34012224	(324. *Ans.*
	3	9	27	
	3	9	7012	
	3	18	5768	
	6	27*	1244224	
	3	184		
1	9*	2884		

[See page 285.]

2	188
92	3072*
2	3856
94	311056
2	
96	
4	
964	

(3) To find the cube root of 9.

1	0	0	9	2.080084. *Ans.*
	2	4	8	
	2	4	1*	
	2	8	.998912	
	4	12*	.001088	
	2	.4864	.001038	
	6*	12.4864	50	
	.08	.4928	51	
	6.08	12.9792*		
	.08			
	6.16			
	.08			
	6.24*			

(4) To find the cube root of 30.

1	0	0	30	3.107233.
	3	9	27	
	3	9	3*	
	3	18	2.791	
	6	27*	.209	
	3	.91	.20223	
1	9*	27.91	.00677	
	.1	.92	579	
	9.1	28.83*	98	
	.1	6	87	
	9.2	28.89	11	
	.1	6	9	
	9.3*	28.95*		

(5) To find the fifth root of 68641485507.

1	0	0	0	0	68641485507 (147
	1	1	1	1	1
	1	1	1	1	586414
	1	2	3	4	437824
	2	3	4	5*	14859085507
	1	3	6	59456	14859085507
	3	6	10*	109456	
	1	4	4864	82624	
	4	10*	14864	192080*	
	1	216	5792	201926501	
1	5*	1216	20656	2122726501	
	4	232	6784		
	54	1448	27440*		
	4	248	1406643		
	58	1696	28846643		
	4	264			
	62	1960*			
	4	4949			
	66	200949			
	4				
1	70*				
	7				
	707				

APPROXIMATION BY DOUBLE POSITION.

(2) $x^3+30x=420$.

6	x	7
216	x^3	343
180	$30x$	210
396	results	553
553	7	420
396	6	396
157 :	1 ::	24 : 0.
6.1	x	6.2
226.981	x^3	238.328

183.0	$30x$	186.0
409.981	results	424.328

14.347 : .1 :: 10.019 : 0.163

By trial x is found to be greater than 6.17, therefore let $x=6.17$ and 6.18.

6.17	x	6.18
234.885	x^3	236.029
185.10	$30x$	185.40
419.985	results	421.429

1.444 : .01 :: .015 : 000103.

$\therefore x=6.17+.000103=6.170103$ nearly.

(3) $144x^3-973x=319$.

2	x	3
1152	$144x^3$	3888
—1946	$-973x$	—2919
— 794	results	+ 969
	3	
+ 969	2	319
1763	1	650

1763 : 1 :: 650 : .3

$x=3-.3=2.7$.

2.7	x	2.8
2834.352	$144x^3$	3161.088
—2627.1	$-973x$	—2724.4
+ 207.252	results	+ 436.688

As 229.436 : .1 :: 111.748 : .048

$\therefore x=2.7+.048=2.748$, and by trial 2.75 is found to verify the equation exactly, hence $x=2.75$.

In the application of the rule of Double Position to the solution of equations, the first correction is generally too small, as in the two preceding solutions, and as may be seen more particularly in the solution of example 5.

To see the reason of this, it must be noticed that the sums, or the differences, of the higher powers of numbers, increase very rapidly as the numbers increase. Hence if two numbers equally distant from the true number, are substituted in any equation containing the second or higher powers of the unknown quantity the result, arising from the substitution of the greater number, will be farther from the true result than that obtained by the sub stitution of the smaller. And hence, by the operation of the rule, the correction will give for the true number a number too small. To illustrate this by an example, suppose we have the equation

$$x^3-x=24, \text{ of which the root is } 3.$$

Let us notice the results obtained by the substitution of 2 and 4 for x.

2	x	4
8	x^3	64
-2	$-x$	-4
6	results	60
-18	errors	$+36$

Difference of the errors $=36-(-18)=54$,

$$\text{then } 54 : 2 :: 18 : \tfrac{2}{3}.$$

Now the true correction is 1, but we obtained $\frac{2}{3}$ because although the suppositions, 2 and 4, are *equally* distant from the true number, yet the corresponding results are *unequally* distant from it. Now the rule proceeds on the hypothesis that the errors of the results are proportional to the errors of the suppositions. But this is never exactly true, and is only nearly so when each of the suppositions is very near the true number. Attention to this principle will often guide the pupil in selecting trial numbers for the second operation.

(4) $x^3+10x^2+5x=2600$.

By trial we find that 11 is so near the true number that we may at once make trial of 11 and 11.1.

11	x	11.1
1331	x^3	1367.631
1210	$10x^2$	1232.1
55	$5x$	55.5
2596	results	2655.231
-4	errors	$+55.231$

59.231 : .1 :: 4 : .006

$\therefore x=11+.006=11.006$ nearly.

11.006	x	11.007
1333.179188	x^3	1333.542617
1211.32036	$10x^2$	1211.54049

55.030	$5x$	55.035
2599.529548	results	2600.118107
.470452	errors	+.118107

.588559 : .001 : .470452 : .00079

$\therefore x=11.006+.00079=11.00679.$

(5) $2x^3+3x^2-4x=10.$

1	x	2
2	$2x^3$	16
3	$+3x^2$	12
—4	$-4x$	— 8
+1	results	+20
—9	errors	+10

19 : 1 :: 1 : .5 nearly.

$\therefore x=1+.5=1.5$ nearly.

By trial, however, we find that 1.6 is too small, and 1.7 too great, let these therefore be the next two assumed numbers.

1.6	x	1.7
+8.192	$2x^3$	+ 9.826
+7.68	$+3x^2$	+ 8.67
—6.4	$-4x$	— 6.8
+9.472	results	+11.696
— .528	errors	+ 1.696

2.224 : .1 :: .528 : .024 nearly.

$\therefore x=1.6+.024=1.624.$

By trial we find 1.624 is too small, and 1.625 too great; using these as the next two assumed numbers, we readily find the next two figures of the root.

(6) $x^4-x^3+2x^2+x=4.$

It is easily seen by inspection, that x is a little more than 1 and by trial it is found greater than 1.1, and less than 1.2; let these, therefore, be the two assumed numbers.

1.1	x	1.2
+1.4641	$+x^4$	+2.0736
—1.331	$-x^3$	—1.728
+2.42	$+2x^2$	+2.88
+1.1	$+x$	+1.2
+3.6531	results	+4.4256
— .3469	errors	+ .4256

7725 : .1 :: .3469 : .045 nearly.

By trial x is found greater than 1.146, and less than 1.147. By repeating the operation with these numbers, we readily find the next two figures of the root.

(7) $x^4+x^3+2x^2-x=4$.

It is easily seen that x is a little more than 1, and by trial it is found less than 1.1; therefore let 1 and 1.1 be the two assumed numbers

1	x	1.1
1	$+x^4$	1.4641
1	$+x^3$	1.331
2	$+2x^2$	2.42
—1	$-x$	—1.1
+3	results	+4.1151
—1	errors.	+ .1151

1.1151 : .1 :: 1 : .09 nearly.

By trial x is found to be greater than 1.09, therefore let 1.09 and 1.1 be the next two assumed numbers.

1.09	x	1.1
1.411581	x^4	1.4641
1.295029	$+x^3$	1.331
2.3762	$+2x^2$	2.42
—1.09	$-x$	1.1
+3.99281	results	+4.1151
— .00719	errors	+ .1151

.12229 : .01 :: .00719 : .00059 nearly.

$\therefore x=1.09+.00059=1.09059$ nearly

(8) $x^4-12x+7=0$.

It is easily seen that x is a little greater than 2. and by trial it is found less than 2.1, therefore let these be the first two assumed numbers.

2.	x	2.1
+16.	x^4	+19.4481
—24.	$-12x$	—25.2
+ 7.	+ 7	+ 7.
— 1.	results and errors	+1.2481

2.2481 : .1 : 1 .04

$\therefore x=2+.04=2.04$ nearly.

By trial we find 2.04 too small, and 2.05 too great.

2.04	x	2.05
17.3189	x^4	17.661
—24.48	$-12x$	—24.6
+ 7	7	+ 7
— .1611	results and errors	+ .061

As .2221 : .01 : .1611 : .0072

$\therefore x=2.04+.0072=2.0472$ nearly.

By trial we find that 2.0472 is too small, and 2.0473 too great; then by using these as the next two assumed numbers we readily find the remaining figure of the root.

(9) $2x^4-13x^2+10x-19=0$.

Here it is readily found that x lies between 2 and 3, let these therefore be the two assumed numbers.

2.	x	3
32.	$2x^4$	162
—52.	$-13x^2$	—117
+20.	$+10x$	+ 30
—19.	—19	— 19
—19.	results and errors	+56

75 : 1 : : 19 : .3 nearly, and $x=2.3$ nearly.

By trial we find that x is greater than 2.4, and less than 2.5 let these therefore be the two assumed numbers.

2.4	x	2.5
66.3552.	$2x^4$	78.125
—74.88	$-13x^2$	—81.25
+24.	$+10x$	+25.
—19.	—19	—19.
— 3.5248.	results and errors	2.875

6.3998 .1 : : 3.5248 : .05.

$\therefore x=2.4+.05=2.45$ nearly.

By trial it is found 2.45 is too small, and 2.46 too great, using these as the next two assumed numbers, we obtain the next two figures of the root.

(10) $\sqrt[3]{7x^3+4x^2}+\sqrt{10x(2x-1)}=28$.

By trial we readily find that x lies between 4 and 5 ; we therefore take these as the first two assumed numbers.

4	x	5
$+8$	$\sqrt[3]{7x^3+4x^2}$	9.91
$+16.73$	$+\sqrt{10x(2x-1)}$	21.21
-28	-28	-28
-3.27	errors	$+3.12$

$$6.39 : 1 :: 3.27 : .51.$$

$$\therefore x=4+.51=4.51 \text{ nearly.}$$

By trial we find x greater than 4.51, and less than 4.52 ; therefore let these be the next two assumed numbers.

4.51	x	4.52
$+8.9773$	$\sqrt[3]{7x^3+4x^2}$	8.9965
$+19.0185$	$+\sqrt{10x(2x-1)}$	$+19.0633$
$-28.$	-28	$-28.$
$-.0042$	errors	$+.0598$

$$.064 : .01 :: .0042 : .00066 \text{ nearly}$$

$$\therefore x=4.51+.00066=4.51066 \text{ nearly.}$$

Article 437.

NEWTON'S METHOD OF APPROXIMATION.

The learner must observe that A is what the proposed equation becomes when $x=a$, and that A' is what the first derived polynomial, or first derived function (Art. 411) becomes when $x=a$.

(1) Proposed equation $X=x^3-2x-5=0$;

First Derived function $X'=3x^2-2$.

When 2 is substituted for x the result is -1, and when 3 is substituted the result is $+16$; therefore (Art. 403), one real root of the equation lies between 2 and 3, and is not much greater than 2. By trial we find that 2.1 gives a positive result, therefore the root lies between 2.0 and 2.1.

$\therefore$ let $x=a+y=2+y$.

then $A=(2)^3-2(2)-5$, and $A'=3(2)^2-2$,

$$y=-\frac{A}{A'}=-\frac{8-4-5}{12-2}=+.1.$$

$\therefore\ x=a+y=2+\left(-\frac{A}{A'}\right)=2+.1=2.1.$

Next let $x=b+z=2.1+z$,

then $B=(2.1)^3-2(2.1)-5$, and $B'=3(2.1)^2-2.$

$$z=-\frac{B}{B'}=-\frac{(2.1)^3-2(2.1)-5}{3(2.1)^2-2}=-\frac{.061}{11.23}=-.0054,$$

$\therefore\ x=b+z=2.1+(-.0054)=2.0946.$

Next let $x=c+z'=2.0946+z$,

then $C=(2.0946)^3-2(2.0946)-5=.000541550536.$

$C'=3(2.0946)^2-2=11.16204748,$

$$z=-\frac{C}{C'}=\frac{.000541550536}{11.16204748}=-.00004851.$$

$\therefore\ x=C+z'=2.0946+(-.00004851)=2.09455149,$

which is true to the seventh place of decimals; and by proceeding in a similar manner the value of x may be found to any required degree of accuracy.

REMARK.— The great objection to Newton's Method of Approximation is, that we are obliged after each operation to commence with the entire approximate value of x in the same manner as at first, and no assistance is derived from the previous calculations except in having found a nearer value of the root. But in Horner's method we approximate continuously to the true value of the root by the evolution of single figures as in Long Division, and the Extraction of the square root in arithmetic, and each previous figure is of use in finding the next. Newton's method is now rarely used, and may be classed among the scientific curiosities of a past age.

Articles 438—441.

CARDAN'S SOLUTION OF CUBIC EQUATIONS.

Formulæ. $x^3+3qx+2r=0.$

$$x=\sqrt[3]{(-r+\sqrt{r^2+q^3})}+\sqrt[3]{(-r-\sqrt{r^2+q^3})}.$$

(2) $x^3-9x+28=0.$ Here $q=-3$, and $r=+14.$

$$x=\sqrt[3]{(-14+\sqrt{196-27})}+\sqrt[3]{(-14-\sqrt{196-27})}$$

$$=\sqrt[3]{(-14+13)}+\sqrt[3]{(-14-13)}=\sqrt[3]{-1}+\sqrt[3]{-27}=-1$$
$$-3=-4.$$

Dividing the given equation by $x-(-4)=x+4$, the quotient is x^2-4x+7; hence $x^2-4x+7=0$, and

$x=2\pm\sqrt{-3}$.

(3) $x^3+6x-2=0$. Here $q=+2$, and $r=-1$.

$x=\sqrt[3]{(+1+\sqrt{1+8})}+\sqrt[3]{(+1-\sqrt{1+8})}=\sqrt[3]{4}+\sqrt[3]{-2}$.

$=\sqrt[3]{4}-\sqrt[3]{2}=1.58740-1.25992=.32748$.

(4) $x^3-6x^2+13x-10=0$.

To remove the second term (see Art. 407, Cor.,)

$$r=-\frac{A}{n}=-\frac{-6}{3}=+2.$$

$\therefore x=y+2$, and the transformed equation is

$(y+2)^3-6(y+2)^2+13(y+2)-10=0$;

$y^3+y=0$, or $y(y^2+1)=0$,

whence $y=0$ and $y^2+1=0$; or $y=\pm\sqrt{-1}$.

$\therefore x=y+2=2$ or $2\pm\sqrt{-1}$.

(5) $x^3+6x^2-32=0$.

Let $x=y-\frac{6}{3}=y-2$.

$(y-2)^3+6(y-2)^2-32=0$;

or $y^3-12y-16=0$.

Here $q=-4$, and $r=-8$.

$y=\sqrt[3]{(+8+\sqrt{64-64})}+\sqrt[3]{(+8-\sqrt{64-64})}$

$=\sqrt[3]{+8}+\sqrt[3]{+8}=+2+2=+4$.

Dividing the equation $y^3-12y-16=0$ by $y-4$ the quotient is y^2+4y+4, hence $y^2+4y+4=0$,

and $y=-2$, and -2.

$\therefore x=y-2=4-2=2$, and $-2-2=-4$, and

$-2-2=-4$.

(6) $x^3+6x^2+27x-26=0$.

Let $x=y-\frac{6}{3}=y-2$, then

$(y-2)^3+6(y-2)^2+27(y-2)-26=0$.

or, $y^3+15y-64=0$.

Here $q=+5$, and $r=-32$.

$y=\sqrt[3]{(32+\sqrt{1024+125})}+\sqrt[3]{(32-\sqrt{1024+125})}$

$=\sqrt[3]{(32+33.896902513356585455)}$

$+\sqrt[3]{(32-33.896902513356585455)}$

$=\sqrt[3]{(65.896902513356585455)}$

$-\sqrt[3]{(1.896902513356585455)}$

$=4.0391346-1.2378889=2.801245+$.

$\therefore\ x=y-2=2.801245-2=.801245$.

(7) $x^3-9x^2+6x-2=0$.

Let $x=y+\frac{9}{3}=y+3$, then

$(y+3)^3-9(y+3)^2+6(y+2)-2=0$

or, $y^3-21y-38=0$.

Here $q=-7$, and $r=-19$.

$x=\sqrt[3]{(19+\sqrt{361-343})}+\sqrt[3]{(19-\sqrt{361-343})}$

$=\sqrt[3]{(19+3\sqrt{2})}+\sqrt[3]{(19-3\sqrt{2})}$

$=\sqrt[3]{(19+3\times1.414213562373095048 8)}$

$+\sqrt[3]{(19-3\times1.4142135623730950488)}$

$=\sqrt[3]{(23.2426406871192851 46)}$

$+\sqrt[3]{(14.7573593128807148 54)}$

$=2.8538325+2.4528418=5.306674+$.

$\therefore\ x=y+3=5.306674+3=8.306674+$.

REMARK.—In the solutions to the last two examples, the extraction of the square root is carried to eighteen places of decimals, but this is further than is necessary to insure accuracy in extracting the cube root to seven places. For this purpose ten places, or even less, are quite sufficient.

After the pupil has faithfully performed all the operations in these two examples, let him solve the same equations by Horner's method, (Art. 434,) and he will then appreciate its superiority. To obtain the result true to six places of decimals requires about one-fourth as much labor by Horner's Method as by Cardan's Rule, and the difference increases rapidly with the increase in the number of places of decimals.

Article 442.

RECIPROCAL OR RECURRING EQUATIONS.

(1) $x^4-10x^3+26x^2-10x+1=0$,

$x^2-10x+26-\frac{10}{x}+\frac{1}{x^2}=0$, by dividing by x^2;

or, $x^2+\frac{1}{x^2}-10\left(x+\frac{1}{x}\right)=-26$;

Let $x+\frac{1}{x}=z$, then $x^2+\frac{1}{x^2}=z^2-2$, and

$z^2-2-10z=-26$,

$z^2-10z=-24$, and $z=6$ or 4.

$\therefore\ x+\frac{1}{x}=6$ or 4,

whence $x=3\pm2\sqrt{2}$, or $2\pm\sqrt{3}$.

(2) $x^4+5x^3+2x^2+5x+1=0$.

$x^2+5x+2+\frac{5}{x}+\frac{1}{x^2}=0$, by dividing by x^2,

$x^2+\frac{1}{x^2}+5\left(x+\frac{1}{x}\right)=-2$.

Let $x+\frac{1}{x}=z$, then $x^2+\frac{1}{x^2}=z^2-2$, and

$z^2+5z=0$, whence $z=0$, or -5.

$\therefore\ x+\frac{1}{x}=0$, or -5,

whence $x=\pm\sqrt{-1}$, or $\frac{1}{2}(-5\pm\sqrt{21})$.

(3) $x^4-\frac{5}{2}x^3+2x^2-\frac{5}{2}x+1=0$

$x^2-\frac{5}{2}x+2-\frac{5}{2x}+\frac{1}{x^2}=0$, by dividing by x^2

$x^2+\frac{1}{x^2}-\frac{5}{2}\left(x+\frac{1}{x}\right)=-2$.

Let $x+\frac{1}{x}=z$, then $x^2+\frac{1}{x^2}=z^2-2$, and

$z^2-\frac{5}{2}z=0$, whence $z=0$, or $+\frac{5}{2}$.

$\therefore\ x+\frac{1}{x}=0$, or $+\frac{5}{2}$;

whence $x=\pm\sqrt{-1}$, or 2, or $\frac{1}{2}$.

(4) $x^4-3x^3+3x-1=0$.

It is proved in Art. 412, Prop. III, that this equation is divisible by x^2-1, $\therefore\ x^2-1=0$, and $x=\pm1$.

Dividing the given equation by x^2-1 the quotient is x^2-3x+1, therefore $x^2-3x+1=0$, whence $x=\frac{1}{2}(3\pm\sqrt{5})$.

(5) $x^5-11x^4+17x^3+17x^2-11x+1=0$.

It follows from Art. 442, Prop. II, that -1 is a root of this equation, therefore it is divisible by $x+1$ (Art. 395).

$$\begin{array}{l} 1-11+17+17-11+1 \quad (-1 \\ \underline{\quad -1+12-29+12-1 \quad} \\ 1-12+29-12+\ 1 \quad 0 \end{array}$$

$\therefore\ x^4-12x^3+29x^2-12x+1=0.$

$x^2-12x+29-\frac{12}{x}+\frac{1}{x^2}=0$, by dividing by x^2

$x^2+\frac{1}{x^2}-12\left(x+\frac{1}{x}\right)=-29.$

Let $x+\frac{1}{x}=z$, then $x^2+\frac{1}{x^2}=z^2-2$, and

$z^2-12z=-27$, whence $z=9$ or 3.

$\therefore\ x+\frac{1}{x}=9$ or 3;

whence $x=\frac{9\pm\sqrt{77}}{2}$, or $\frac{3\pm\sqrt{5}}{2}$.

(6) $4x^6-24x^5+57x^4-73x^3+57x^2-24x+4=0$.

$4x^3-24x^2+57x-73+\frac{57}{x}-\frac{24}{x^2}+\frac{4}{x^3}=0$, by dividing by x^3,

$4\left(x^3+\frac{1}{x^3}\right)-24\left(x^2+\frac{1}{x^2}\right)+57\left(x+\frac{1}{x}\right)=73.$

Let $x+\frac{1}{x}=z$, then $x^2+\frac{1}{x^2}=z^2-2$.

and $x^3+\frac{1}{x^3}=z^3-3z$.

$\therefore\ 4(z^3-3z)-24(z^2-2)+57z=73,$

$z^3-6z^2+\frac{45}{4}z=+\frac{25}{4}.$

To solve this equation by Cardan's Rule, Art. 441,

let $z=y+2$, then $y^3-\frac{3}{4}y+\frac{1}{4}=0$,

$y=\sqrt[3]{\left(-\frac{1}{8}+\sqrt{\frac{1}{64}-\frac{1}{64}}\right)}+\sqrt[3]{\left(-\frac{1}{8}-\sqrt{\frac{1}{64}-\frac{1}{64}}\right)}$

$=-\frac{1}{2}-\frac{1}{2}=-1.$

Dividing $y^3-\frac{3}{4}y+\frac{1}{4}$ by $y+1$, the quotient is $y^2-y+\frac{1}{4}$, therefore, $y^2-y+\frac{1}{4}=0$, and $y=+\frac{1}{2}$, and $+\frac{1}{2}$.

$\therefore\ z=y+2=-1+2=1$, or $\frac{1}{2}+2=\frac{5}{2}$, and $\frac{5}{2}$.

$\therefore\ x+\frac{1}{x}=1$, whence $x=\frac{1\pm\sqrt{-3}}{2}$;

or, $x+\frac{1}{x}=\frac{5}{2}$, whence $x=2$, or $\frac{1}{2}$.

$\therefore$ the six roots are $2, \frac{1}{2}, 2, \frac{1}{2}, \frac{1+\sqrt{-3}}{2}$, and $\frac{1-\sqrt{-3}}{2}$.

Article 443.

BINOMIAL EQUATIONS.

(1) Let $x^4=1$, then $x^4-1=0$, and $(x^2-1)(x^2+1)=0$
$\therefore\ x^2-1=0$, whence $x^2=1$, and $x=+1$, or -1.
Also, $x^2+1=0$, whence $x^2=-1$, and $x=+\sqrt{-1}$, or, $-\sqrt{-1}$.

(2) Let $x^5=1$, then $x^5-1=0$, and the equation is divisible by $x-1$, $\therefore\ x-1=0$, and $x=+1$.
Dividing x^5-1 by $x-1$, and placing the quotient equal to zero, we have

$x^4+x^3+x^2+x+1=0$,

$x^2+x+1+\frac{1}{x}+\frac{1}{x^2}=0$, by dividing by x^2,

$x^2+\frac{1}{x^2}+x+\frac{1}{x}=-1$.

Let $x+\frac{1}{x}=z$, then $x^2+\frac{1}{x^2}=z^2-2$, and

$z^2+z=1$; whence $z=\frac{-1\pm\sqrt{5}}{2}=a$.

$\therefore\ x+\frac{1}{x}=a$, whence

$x=\frac{a}{2}+\frac{1}{2}\sqrt{a^2-4}$, or $\frac{a}{2}-\frac{1}{2}\sqrt{a^2-4}$, and since a has two values x will have four values.

$$a^2=\frac{(-1\pm\sqrt{5})^2}{4}=\frac{6-2\sqrt{5}}{4}, \text{ or } \frac{6+2\sqrt{5}}{4}.$$

$$x=\frac{a}{2}+\tfrac{1}{2}\sqrt{a^2-4}=\frac{-1+\sqrt{5}}{4}+\tfrac{1}{4}\sqrt{-10-2\sqrt{5}}$$

$$=\tfrac{1}{4}\{\sqrt{5}-1+\sqrt{(-10-2\sqrt{5})}\};$$

$$x=\frac{a}{2}-\tfrac{1}{2}\sqrt{a^2-4}=\frac{-1+\sqrt{5}}{4}-\tfrac{1}{4}\sqrt{-10-2\sqrt{5}}$$

$$=\tfrac{1}{4}\{\sqrt{5}-1-\sqrt{(-10-2\sqrt{5})}\};$$

$$x=-\frac{a}{2}+\tfrac{1}{2}\sqrt{a^2-4}=\frac{-1-\sqrt{5}}{4}+\tfrac{1}{4}\sqrt{-10+2\sqrt{5}}$$

$$=-\tfrac{1}{4}\{\sqrt{5}+1-\sqrt{(-10+2\sqrt{5})}\};$$

$$x=\frac{a}{2}-\tfrac{1}{2}\sqrt{a^2-4}=\frac{-1-\sqrt{5}}{4}-\tfrac{1}{4}\sqrt{-10+2\sqrt{5}}$$

$$=-\tfrac{1}{4}\{\sqrt{5}+1+\sqrt{(-10+2\sqrt{5})}\}.$$

APPENDIX.

INDETERMINATE ANALYSIS.

ART. 1. Indeterminate Analysis is the resolution of equations where the number of unknown quantities is greater than the number of independent equations, and where the results are required in *positive integers.*

It is shown (Alg. Part II, Art. 168,) that whenever the number of unknown quantities is greater than the number of independent equations, an unlimited number of values may be found for each of the unknown quantities. But such conditions may exist as to limit the number of results, or even render the question impossible. Thus the equation $3x+5y=42$, may be satisfied by an infinite number of values of x and y; but if it be required that these values shall be *integral* and *positive,* then we can only find $x=9$ or 4, and $y=3$ or 6.

Problems of this kind are called *indeterminate,* and the results are generally required in *positive integers.*

An indeterminate equation of the *first degree,* containing two unknown quantities, is of the form

$$ax+by=c,$$

where a, b, and c are either positive or negative whole numbers.

ART. 2. PROPOSITION I.—*If an equation of the form $ax+by=c$, is in its lowest terms, it can not be solved unless a and b are prime to each other.*

For, if possible, let $a=md$, and $b=nd$; then $mdx+ndy=c$,

$$\text{and } \therefore\ mx+ny=\frac{c}{d};$$

but, by hypothesis, a, b, and c contain no common factor, therefore $\frac{c}{d}$ is a fraction, and we have the sum of two whole numbers equal to a fraction, which is absurd; hence the proposition is true.

ART. 3. PROPOSITION II.—*If a and b are prime to each other, each term of the series, b, $2b$, $3b$, &c.. . . $(a-1)b$, when divided by a, will leave a different positive remainder.*

For, if possible, let any two of the terms, as mb and nb, when divided by a, leave the same remainder r, so that

$$\frac{mb}{a}=p+\frac{r}{a}, \text{ and } \frac{nb}{a}=q+\frac{r}{a};$$

Subtracting the second equation from the first,

$$\frac{mb}{a}-\frac{nb}{a}=p-q, \text{ or } \frac{b}{a}(m-n)=p-q;$$

but the left hand member of this equality is a fraction, since $\frac{b}{a}$ is a fraction in its lowest terms, and $m-n$ less than a, each being less than a, therefore we have a fraction equal to a whole number, which is absurd; hence the remainders are all different.

Illustration.—Let $a=4$, and $b=7$; then $7, 7\times2$, and 7×3, when divided by 4, leave the different remainders 3, 2, and 1.

Cor. Since the remainders are all different, and are $a-1$ in number, each being less than a, therefore they include all numbers from 1 to $a-1$.

ART. 4. PROPOSITION III.—*The equation* $ax-by=\pm1$ *is always possible in integers, if a and b are prime to each other.*

By the Corollary to the preceding proposition, if b, or some multiple of b less than ab, be divided by a, the remainder will be 1; let y be that multiple, then

$\frac{y\times b}{a}=x+\frac{1}{a}$; x being the integral part of the quotient and 1 the remainder. By clearing and transposing this gives $ax-by=-1$, which proves part of the proposition. Again, by the same corollary, y may be some coëfficient of b less than a, such that $y\times b$, when divided by a, will leave a remainder $a-1$, that is

$\frac{y\times b}{a}=x+\frac{a-1}{a}$, x being the integral part of the quotient, and $a-1$ the remainder; by clearing this gives

$by=ax+a-1$; by transposing and factoring

$a(x+1)-by=1$; let $x+1=x'$, this gives

$ax'-by=1$, which proves the remaining part of the proposition. Hence, if a and b are prime to each other, such values of x and y may always be found as will satisfy the equation $ax-by=\pm1$.

ART. 5. PROPOSITION IV.— *If a and b are prime to each other, the equation $ax-by=\pm c$, is always possible, and an indefinite number of integral values may be assigned to x and y, which will satisfy the equation.*

For $ax'-by'=\pm 1$, is always possible (Art. 4).

$\therefore$ $c(ax'-by')=\pm c$, or $acx'-bcy'=\pm c$, is always possible.

Let $cx'=x$ and $cy'=y$, then

$ax-by=\pm c$, is always possible.

Let one solution be $x=p$ and $y=q$, then

$$ax-by=ap-bp, \text{ or } ax-ap=by-bq.$$

$$\therefore \frac{a(x-p)}{b(y-q)}=1, \text{ and } \frac{x-p}{y-q}=\frac{b}{a}=\frac{mb}{ma},$$

$$\text{or, } x-p=mb, \text{ and } y-q=ma;$$

$$\therefore x=p+mb, \text{ and } y=q+ma.$$

and since m may be either positive or negative, and have any value whatever from 0 to infinity, the number of values of x and y are indefinite.

Cor. Since p and q are integers, and since m may be either positive or negative, m may be so assumed, that x shall be less than b, or that y shall be less than a; for making m equal to 0, -1, -2, -3, &c., successively, we shall have

$$x=p, p-b, p-2b, \text{ \&c.},$$

$$\text{and } y=q, q-a, q-2a, \text{ \&c.},$$

where it is obvious that one of the values of x must be less than b, and one of the values of y less than a, whatever be the values of p and q.

ART. 6. PROPOSITION V. — *The equation $ax+by=c$, is always possible in positive whole numbers, provided a and b are prime to each other, and c is greater than $ab-a-b$.*

For, if $c=(ab-a-b)+r$, the equation becomes

$$ax+by=ab-a-b+r;$$

$$\therefore x=\frac{ab-a-b-by+r}{a}=b-1-\frac{(y+1)b-r}{a};$$

Since $b-1$ is an integer, the possibility depends on

$$\frac{(y+1)b-r}{a}=z, \text{ being an integer.}$$

Let $y+1=y'$, then we have $az-by'=-r$, which is always possible, (Art. 4); let then y' be less than a, or $y+1<a$ (Prop. IV, Cor.), then in the equation $\frac{(y+1)b-r}{a}=z$,

z must be less than $b-1$, therefore

$x=b-1-\frac{(y+1)b-r}{a}=b-1-z$, must be some integer number ; hence the equation $ax+by=c$ is always possible when a and b are prime to each other, and $c>(ab-a-b)$.

REMARK.— The last two propositions are of great practical utility, inasmuch as they show the possibility or impossibility of equations of this kind.

ART. 7. PROBLEM I.— *To find positive integral values of x and y in the equation*

$$ax-by=c,$$

$$\text{or, } ax+by=c,$$

a and b being prime to each other, and c being either plus or minus.

$ax-by=c$, gives $x=\frac{by+c}{a}=py+q+\frac{b'y+c'}{a}$, where $py+q$ represents the integral part of the quotient, and $b'y+c$ the remainder, b' and c' being less than a. Now in order that the value of x shall be integral, the remainder $b'y+c'$ must be divisible by a, hence $\frac{b'y+c'}{a}$ must be a whole number.

If now we take the difference between $\frac{ay}{a}$, which is evidently a whole number, and that multiple of $\frac{b'y+c'}{a}$ in which $p'b'y$, the multiple of $b'y$, is nearest to ay, we shall have a remainder of the form $\frac{b''y+c''}{a}$, in which b'' is less than b'. Again, if we take the difference of $\frac{b'y+c'}{a}$, and that multiple of $\frac{b''y+c''}{a}$, in which the multiple of $b''y$ is nearest to $b'y$, we shall have a remainder of the form $\frac{b'''y+c'''}{a}$, which must be a whole number. Hence, by continuing this process, we shall finally obtain a remainder of the form $\frac{y+k}{a}$, or $\frac{y-k}{a}$, in which the coëfficient of y is 1.

Now if we divide k by a, and call the quotient q, and the remainder r, we shall have $\frac{y+k}{a}=q+\frac{y+r}{a}$, or $\frac{y-k}{a}=-q+\frac{y-r}{a}$, which are evidently whole numbers when $\frac{y+r}{a}$, or $\frac{y-r}{a}$ are whole numbers. Now let $\frac{y+r}{a}=w$, a whole number; then

$y=aw-r$, where w may be any whole number that will render y positive. In a similar manner, if r is negative, we find $y=aw+r$.

It is evident that the same general method may be applied to find the value of y in the equation $ax+by=c$.

Since the subtraction of fractions does not produce any change in the common denominator, this may be omitted in the operation, and we may proceed according to the following

RULE.—*Reduce the equation to the form $x=by+c$; perform the division of $by+c$ by a, and call the remainder $b'y+c'$.*

Take the difference between ay and that multiple of $b'y+c'$ in which the multiple of $b'y$ is the nearest to ay, and call the remainder $b''y+c''$.

Again, take the difference between $b'y+c'$, and that multiple of $b''y+c''$, in which the multiple of $b''y$ is the nearest to $b'y$. And so on, till we get a remainder of the form $y+k$, or $y-k$. Lastly divide k by a and call the remainder r; then $y=aw-r$, or $aw+r$, according as k is plus or minus; and w may be any whole number that will render y positive.

Having the value of y, the general value of x is obtained by substituting the value of y in the given equation.

When the given equation is of the form $ax+by=c$, the value of y is found on the same principles, except that it may be necessary to add instead of subtracting, to reduce the coëfficient of y.

The preceding rule depends on the principle that *the sum or difference of two whole numbers is a whole number; and that any multiple of a whole number is also a whole number.*

EXAMPLES.

1. Given $7x-12y=15$, to find x and y in positive whole numbers.

Here $x=\frac{12y+15}{7}=y+2+\frac{5y+1}{7}$, and $a=7$.

$$7y \quad = ay$$
$$\underline{5y+1=b'y+c'}, \text{ and the multiple } =1,$$
$$\underline{2y-1=b''y+c''},$$
$$4y-2=pb''y+pc'', \text{ where } p=2.$$
$$\underline{5y+1=b'y+c'},$$
$$y+3= \text{ diff. of last two quantities},$$
$$\frac{y+3}{7}=w, \text{ and } y=7w-3,$$
$$x=\frac{12(7w-3)+15}{7}=12w-3.$$

Let $w=1, 2, 3,$ &c.

Then $x=9, 21, 33,$ &c.,

and $y=4, 11, 18,$ &c., where it is obvious the number of values of x and y are unlimited.

2. Given $7x+11y=47$, to find x and y in positive whole numbers.

Here $x=\frac{47-11y}{7}=6-y+\frac{5-4y}{7}$, and $a=7$.

$$5-4y=b'y+c',$$
$$10-8y=pb'y+pc',$$
$$\underline{7y=ay},$$
$$10-y,$$
$$\frac{10-y}{7}=1+\frac{3-y}{7};$$
$$\frac{3-y}{7}=w, \therefore y=3-7w;$$
$$x=\frac{47-11(3-7w)}{7}=2+11w.$$

Let $w=0$, then $x=2$, and $y=3$, the only values.

3. How can 78 francs be paid with pieces of 5 francs and of 3 francs, and in how many ways?

Let $x=$ the number of 5 franc pieces, and $y=$ the number of 3 franc pieces.

Then $5x+3y=78$,

$$x=\frac{78-3y}{5}=15+\frac{3-3y}{5},$$
$$3-3y=b'y+c',$$

$$6-6y=pb'y+pc',$$
$$5y=ay,$$
$$\overline{6-y}$$
$$\frac{6-y}{5}=1+\frac{1-y}{5}; \text{ let } \frac{1-y}{5}=w, \therefore y=1-5w;$$
$$x=\frac{78-3(1-5w)}{5}=15+3w.$$

Let $w=0, -1, -2,$ &c.

Then $x=15, 12, 9, 6, 3, 0.$

$y=1, 6, 11, 16, 21, 26.$ Hence it may be paid in 6 ways.

EXAMPLES FOR PRACTICE.

4. Given $5x+7y=19$, to find x and y.
Ans. $x=1$, $y=2$, only one solution.

5. Given $7x+19y=92$, to find x and y.
Ans. $x=5$, $y=3$, only one solution.

6. $5x+7y=29$. $x=3$, $y=2$.

7. $13x+14y=200$. $x=10$, $y=5$.

8. $27x+16y=1600$. $x=48, 32, 16$.
$y=19, 46, 73$.

9. A owes B £100, but A has no money but guineas, and B has only 50 crowns; how can the debt be paid, a guinea being 21 shillings, and a crown 5 shillings? *Ans.* A gives B 100 guineas, and receives 20 crowns from B; or A gives B 105 guineas, and receives 41 crowns from B.

10. Find two fractions, whose denominators shall be 7 and 9, and their sum equal to $\frac{19}{21}$.

Let x and y denote the numerators of two fractions, then

$$\frac{x}{7}+\frac{y}{9}=\frac{19}{21}, \text{ or } 9x+7y=57, \text{ whence } x=4, \text{ and } y=3.$$

Ans. $\frac{4}{7}$ and $\frac{3}{9}$.

11. Find two fractions whose denominators are 7 and 9, and whose sum is $1\frac{47}{63}$. *Ans.* $\frac{6}{7}$ and $\frac{8}{9}$.

12. Of the equations $9x+17y=127$, and $9x+17y=128$, which is possible, and which impossible?
Ans. First impossible. Second possible, $x=1$, $y=7$.

ART. 8. PROBLEM II.— To determine the number of solutions of which the equation

$$ax+by=c,$$

will admit in positive whole numbers.

Let m denote an undetermined positive whole number, and let x', y' satisfy the equation

$$ax'-by'=1,$$

then $\quad acx'-bcy'=c,$

and $\quad -abm+abm=0,$

whence, $a(cx'-bm)+b(am-cy')=c;$

but $\quad ax+by=c,$

$\therefore\ x=cx'-bm$, and $y=am-cy'$.

Now it is evident that the number of solutions will be the same as the number of values that can be assigned to m that will render bm less than cx', and am greater than cy',

$$bm<cx', \text{ gives } m<\frac{cx'}{b};$$

$$am>cy', \text{ gives } m>\frac{cy'}{a}.$$

Hence the number of values of m will correspond to the difference between the integral parts of the fractions

$$\frac{cx'}{b}, \text{ and } \frac{cy'}{a},$$

except when $\frac{cx'}{b}$ is a whole number. In this case, since

$m<\frac{cx'}{b}$, the number of solutions will be one less, or, which amounts to the same thing $\frac{b}{b}$ must be considered a fraction.

EXAMPLES.

1. Given $5x+11y=254$, to find the number of values of x and y in whole positive numbers,

$$5x'-11y'=1.$$

$$x'=\frac{11y'+1}{5}=2y'+\frac{y'+1}{5}.\ \therefore\ y'=5w-1.$$

By substituting we find $x'=11w-2$.

Let $w=1$, then $x'=9$, and $y'=4$.

$$\frac{cx'}{b}=\frac{254\times9}{11}=207\tfrac{9}{11},$$

$$\frac{cy'}{a}=\frac{254\times4}{5}=203\tfrac{1}{5}. \qquad \frac{cx'}{b}-\frac{cy'}{a}=4.$$

This result may be verified by actually finding the values of x and y. Thus, $x=9, 20, 31$, or 42; and $y=19, 14, 9$, or 4.

2. Given $7x+9y=2342$; to find the number of values of x and y in positive whole numbers. *Ans.* 37.

3. Given $11x+17y=987$, to find the number of values of x and y in positive integers. *Ans.* 5.

4. Given $9x+13y=2000$, to find the number of solutions in positive integers. *Ans.* 17.

5. In how many ways can £100 be paid in crowns and guineas, the crown being 5s. and the guinea 21s. *Ans.* 19.

6. In how many ways can £1053 be paid in guineas and moidores, the guinea being 21s. and the moidore 27s.?

Ans. 111 ways.

Art. 9. Problem III.—To find the integral values of x, y, and z, in the equation $ax+by+cz=d$.

Let c be the greatest coëfficient in this equation, then since the values of x and y can not be less than 1, the value of z can not exceed

$$\frac{d-a-b}{c}.$$

But $$x=\frac{d-by-cz}{a},$$

therefore, by operating on this equation, according to the method employed in Art. 6, we shall obtain a result of the form

$\frac{y\pm nz\pm r}{a}$; let this equal w, then

$y=aw\pm nz\pm r$, where z may have any value from 1 to $\frac{d-a-b}{c}$, that will give positive integral values to x and y.

EXAMPLES.

1. Given $3x+5y+7z=50$.

Here z can not exceed $\frac{50-3-5}{7}=6$.

$$x=\frac{50-5y-7z}{3}=16-y-2z+\frac{2-2y-z}{3};$$

$$3y=ay,$$

$$\frac{2-2y-z}{y-z+2},\ \therefore\ y=3w+z-2$$

$$x=\frac{50-5(3w+z-2)-7z}{3}=20-5w-4z.$$

If $w=1$, $x=15-4z$, let $z=1, 2, 3,$
$y=1+z$, then $x=11, 7, 3,$
$y=2, 3, 4.$

If $w=2$, $x=10-4z$, let $z=1, 2,$
$y=4+z$, then $x=6, 2,$
$y=5, 6.$

If $w=3$, $x=5-4z$, let $z=1,$
$y=7+z$, then $x=1,$
$y=8.$

Therefore, the whole number of solutions is 6.

When the number of solutions is numerous, the process will become tedious ; but the object of inquiry in such problems is generally not to find the solutions themselves, but to determine the number of which the equation admits, the method of doing which will be explained in the next problem.

2. Given $2x+3y+4z=21$, to find all the positive integral values of x, y, and z.

Ans. $z=1$: $\begin{cases}x=7, 4, 1;\\ y=1, 3, 5.\end{cases}$ | $z=2$: $\begin{cases}x=5, 2;\\ y=1, 3.\end{cases}$ | $z=3$: $\begin{cases}x=3;\\ y=1.\end{cases}$ | $z=4$: $\begin{cases}x=1.\\ y=1\end{cases}$

3. Given $2x+5y+4z=27$, to find all the positive integral values of x, y, and z.

Ans.

$z=1$	2	3	4	5	1	2
$y=1$	1	1	1	1	3	3.
$x=9$	7	5	3	1	4	2

4. Given $17x+19y+21z=400$, to find the integral values of x, y, and z.

Ans. $z=$ 1, 2, 3, 4, 5, 6, 11, 12, 13, 14,
$y=$11, 9, 7, 5, 3, 1, 8, 6, 4, 2,
$x=$10, 11, 12, 13, 14, 15, 1, 2, 3, 4.

ART. **10.** PROBLEM IV.—To determine the number of solutions of which the equation $ax+by+cz=d$

will admit, at least two of the coëfficients, a, b, c, being prime to each other.

By Art. 8 the number of solutions of which the equation $ax+by=c$ will admit, is expressed by the difference between the integral parts of $\frac{cx'}{b}$, and $\frac{cy'}{a}$, where x' and y' are to be found from the equation $ax'-by'=1$. Now in the equation

$ax+by+cz=d$, if we transpose cz, we have

$ax+by=d-cz$; therefore, if we make $z=1$, 2, 3, 4, &c., successively, the number of solutions in the equations

$$\left\{\begin{array}{l} ax+by=d-c \\ ax+by=d-2c \\ ax+by=d-3c, \\ \quad\&c. \end{array}\right. \quad \text{will be the difference of the integral parts of} \quad \begin{array}{l} \frac{(d-c)x'}{b}, \text{ and } \frac{(d-c)y'}{a} \\ \frac{(d-2c)x'}{b}, \text{ and } \frac{(d-2c)y'}{a} \\ \frac{(d-3c)x''}{b}, \text{ and } \frac{(d-3c)y}{a} \\ \quad\&c. \end{array}$$

Now the sum of these differences will be the whole number of solutions of which the equation admits. Therefore, if we take the sum of the integral parts of the arithmetical series

$$\frac{(d-c)x'}{b}+\frac{(d-2c)x'}{b}+, \&c. \dots\dots\dots +\frac{(d-z\,c)x'}{b},$$

and also of the arithmetical series

$$\frac{(d-c)y'}{a}+\frac{(d-2c)y'}{a}+, \&c. \dots\dots\dots +\frac{(d-z'c)y}{a},$$

the difference of the two will be the whole number of integral solutions. Now in each of these series the first and last terms, and also the number of terms are known; for the general term in the first series being $\frac{(d-zc)x'}{b}$, and in the second, $\frac{(d-zc)y'}{a}$ the extreme terms will be found by taking $z=1$, and z the greatest whole number in $\frac{d-a-b}{c}$; the last value of z being found by making x and y, each equal 1 in the given equation. It is also obvious that the last value of z expresses the number of terms in the series.

Therefore, if we find the sums of the terms in each series, and deduct from each the sum of its fractional parts, we shall obtain the sums of the integral parts of each series.

In finding the sums of the fractional parts, since the denominator is the same, it is obvious that the fractions must recur in periods, and that the greatest number of fractions in each period can never exceed the denominator, since any divisor, as b, can leave no other remainder than those from 1 to $b-1$; hence, the shortest method of finding the sum of all the fractions, will be to find the sum of the fractions in one period, and multiply this by the number of periods. If there are not an exact number of periods, the overplus fractions must be summed by themselves, observing that they will recur in the same order as in the first period. Also, in the first series, $\frac{b}{b}$ must be considered as a fraction. (See Art. 8.)

EXAMPLES.

1. Given $3x+5y+7z=100$, to find the number of solutions of which it admits in positive integers.

Here $3x+5y=100-7z$. If we make $z=1, 2, 3, 4 \ldots 13$, in succession, then the number of solutions, of which the equation $3x+5y=d$ will admit, is expressed by $\frac{dx'}{5}-\frac{dy'}{3}$, where x' and y' are to be found from the equation $3x'-5y'=1$, (x' being $=2$, and $y'=1$). Therefore, in the equation $3x+5y=100-7z$, if we take $z=1, 2, 3$, to 13, which is the limit to the value of z, the number of solutions in the equations $3x+5y=9$, $3x+5y=16$, and to $3x+5y=93$, will be expressed by $\frac{9x'}{5}-\frac{9y'}{3}$, $\frac{16x'}{5}-\frac{16y'}{3}$, &c., to $\frac{93x'}{5}-\frac{93y'}{3}$, or by

$$\left\{\frac{9x'}{5}+\frac{16x'}{5} \ldots +\frac{93x'}{5}\right\} - \left\{\frac{9y'}{3}+\frac{16y'}{3} \ldots +\frac{93y'}{3}\right\}.$$

Or, by substituting the values of x' and y' the number of solutions will be expressed by the difference of the arithmetical series,

$$\frac{2.9}{5}+\frac{2.16}{5}+\frac{2.23}{5}+\frac{2.30}{5} \ldots \ldots \frac{2.93}{5},$$

and $$\frac{1.9}{3}+\frac{1.16}{3}+\frac{1.23}{3}+\frac{1.30}{3} \ldots \ldots \frac{1.93}{3}.$$

The sum of the first series is $265\frac{1}{3}$, and of the second 221. But as it is only the sum of the integral numbers in each that is wanted, we must deduct from each the sum of the fractions in it.

The fractions in the first series occur in the following order: $\frac{3}{5}, \frac{2}{5}, \frac{1}{5}, \frac{5}{5}, \frac{4}{5}$; and as there are five terms in this period there will be $(\frac{13}{5}=2\frac{3}{5})$ two such periods, and 3 terms besides. $\frac{3}{5}+\frac{2}{5}+\frac{1}{5}+\frac{5}{5}+\frac{4}{5}=\frac{15}{5}=3$, and $3\times2=6$, $\frac{3}{5}+\frac{2}{5}+\frac{1}{5}=1\frac{1}{5}$, and $6+1\frac{1}{5}=7\frac{1}{5}$, the sum of the fractions in the first series. In the second series $\frac{0}{3}+\frac{1}{3}+\frac{2}{3}=\frac{3}{3}=1$; $\frac{13}{4}=4\frac{1}{3}$, hence there are 4 periods of fractions, and the whole is $1\times4=4$.

$$265\frac{1}{3}-7\frac{1}{5}=258\frac{2}{15}\text{ ; } 221-4=217.$$

$258\frac{2}{5}-217=41\frac{2}{5}$, hence the number of solutions required is 41.

2. Given the equation $2x+3y+5z=41$, to find the number of solutions of which it admits in integers. *Ans.* 21.

3. Given the equation $5x+7y+11z=224$, to find the number of solutions of which it admits in integers. *Ans.* 59.

4. It is required to determine the number of integral solutions of which the equation $17x+21y+30z=3000$ will admit. *Ans.* 40b.

5. It is required to determine the number of integral solutions of which the equation $7x+9y+23z=9999$ will admit. *Ans.* 34365.

ART. 11. In the preceding problem it is required that at least two of the coëfficients shall be prime to each other. When this is not the case, the proposed equation may be easily transformed to another possessing the required property, as is shown in the following example.

Given $12x+15y+20z=601$.

Transposing $20z$, and dividing by 3, we have

$$4x+5y=200-6z+\frac{1-2z}{3};$$

$$\frac{1-2z}{3}+\frac{3z}{3}=\frac{z+1}{3}=u, \text{ hence } z=3u-1\text{; whence}$$

by substitution the proposed equation becomes

$12x+15y+20(3u-1)=601$, which, by reduction, becomes $4x+5y+20u=207$.

Now in this equation x and y have the same values as in the one proposed, and therefore the number of solutions must be the same.

Art. 12. Problem V. *To find the values of three unknown quantities in two equations.*

If two equations, containing three unknown quantities be given one of the unknown quantities may be eliminated, and the value of the other unknown quantities found as in Art. 7.

EXAMPLES.

1. Given $\left.\begin{matrix}3x+5y+2z=40\\4x+4y+\ z=33\end{matrix}\right\}$ to find all the integral values of x, y, and z.

By eliminating z, we obtain $5x+3y=26$, then by Art. 7, we find $x=1+3w$, and $y=7-5w$, and by substituting these values in either of the equations, we find $z=1+8w$.

By taking $w=0$, we find $x=1$, $y=7$, and $z=1$.
" " $w=1$, we find $x=4$, $y=2$, and $z=9$, which are the only values.

2. Given $\left.\begin{matrix}x-2y+z=5\\2x+\ y-z=7\end{matrix}\right\}$ to find the values of x, y, and z.

Ans. $x=5,\ 6,\ 7,$
$y=3,\ 6,\ 9,$
$z=6, 11, 16,$ &c.

3. Given $\left.\begin{matrix}2x+5y+3z=\ 51\\10x+3y+2z=120\end{matrix}\right\}$ to find all the integral values of x, y, and z.

Ans. $x=10$, $y=2$, and $z=7$.

Art. 13. Problem V. *To find the least whole number, which being divided by given numbers, shall leave given remainders.*

Let x represent the required number; a, b, c, &c., the given divisors; and f, g, h, &c., the respective remainders. Then, by subtracting each of the remainders from x, and dividing by a, b, c &c., we have $\frac{x-f}{a}$, $\frac{x-g}{b}$, $\frac{x-h}{c}$, &c., where it is required to find such a value of x, that each of these expressions shall be whole numbers.

Let $\frac{x-f}{a}=p$, then $x=ap+f$; substituting this value of x in the second expression, we have $\frac{ap+f-g}{b}$, in which it is required to find such a value of p as will render the expression a whole number. This may be done as in Prob. 1, Art. 7.

Having found a value of p, substitute it in the expression $ap+$

Now if we divide k by a, and call the quotient q, and the remainder r, we shall have $\frac{y+k}{a}=q+\frac{y+r}{a}$, or $\frac{y-k}{a}=-q+\frac{y-r}{a}$, which are evidently whole numbers when $\frac{y+r}{a}$, or $\frac{y-r}{a}$ are whole numbers. Now let $\frac{y+r}{a}=w$, a whole number; then

$y=aw-r$, where w may be any whole number that will render y positive. In a similar manner, if r is negative, we find $y=aw+r$.

It is evident that the same general method may be applied to find the value of y in the equation $ax+by=c$.

Since the subtraction of fractions does not produce any change in the common denominator, this may be omitted in the operation, and we may proceed according to the following

RULE.—*Reduce the equation to the form $x=by+c$; perform the division of $by+c$ by a, and call the remainder $b'y+c'$.*

Take the difference between ay and that multiple of $b'y+c'$ in which the multiple of $b'y$ is the nearest to ay, and call the remainder $b''y+c''$.

Again, take the difference between $b'y+c'$, and that multiple of $b''y+c''$, in which the multiple of $b''y$ is the nearest to $b'y$. And so on, till we get a remainder of the form $y+k$, or $y-k$. Lastly divide k by a and call the remainder r; then $y=aw-r$, or $aw+r$, according as k is plus or minus; and w may be any whole number that will render y positive.

Having the value of y, the general value of x is obtained by substituting the value of y in the given equation.

When the given equation is of the form $ax+by=c$, the value of y is found on the same principles, except that it may be necessary to add instead of subtracting, to reduce the coëfficient of y.

The preceding rule depends on the principle that *the sum or difference of two whole numbers is a whole number; and that any multiple of a whole number is also a whole number.*

EXAMPLES.

1. Given $7x-12y=15$, to find x and y in positive whole numbers.

Here $x=\frac{12y+15}{7}=y+2+\frac{5y+1}{7}$, and $a=7$.

$$7y \quad = ay$$
$$\underline{5y+1} = b'y+c', \text{ and the multiple } = 1,$$
$$\underline{2y-1} = b''y+c'',$$
$$4y-2 = pb''y+pc'', \text{ where } p=2.$$
$$\underline{5y+1} = b'y+c',$$
$$y+3 = \text{ diff. of last two quantities,}$$
$$\frac{y+3}{7} = w, \text{ and } y = 7w-3,$$
$$x = \frac{12(7w-3)+15}{7} = 12w-3.$$

Let $w=1, 2, 3,$ &c.

Then $x=9, 21, 33,$ &c.,

and $y=4, 11, 18,$ &c., where it is obvious the number of values of x and y are unlimited.

2. Given $7x+11y=47$, to find x and y in positive whole numbers.

Here $x=\frac{47-11y}{7}=6-y+\frac{5-4y}{7}$, and $a=7$.

$$5-4y = b'y+c',$$
$$10-8y = pb'y+pc',$$
$$\underline{7y = ay,}$$
$$10-y,$$
$$\frac{10-y}{7} = 1+\frac{3-y}{7};$$
$$\frac{3-y}{7} = w, \therefore y = 3-7w;$$
$$x = \frac{47-11(3-7w)}{7} = 2+11w.$$

Let $w=0$, then $x=2$, and $y=3$, the only values.

3. How can 78 francs be paid with pieces of 5 francs and of 3 francs, and in how many ways?

Let $x=$ the number of 5 franc pieces, and $y=$ the number of 3 franc pieces.

Then $5x+3y=78$,

$$x = \frac{78-3y}{5} = 15+\frac{3-3y}{5},$$
$$3-3y = b'y+c',$$

3. Find two numbers whose sum and product are equal.

Ans. a and $\frac{a}{a-1}$, where a may be any number whatever.

4. Find the number of solutions in the equation $9x+13y=2000$. *Ans.* 17.

5. What formula gives numbers which, when divided by 3, 4, 5, respectively, leaves the remainders 2, 3, 4?

Ans. $x=60p-1$.

6. Divide 1591 into two such parts that the one may be divisible by 23, and the other by 34.

Ans. 1081, and 510; or 299, and 1292.

7. Into how many pairs of numbers may 350 be divided, such that one number, when divided by 5, shall leave a remainder 3, and if 5 be taken from the other number it shall be a multiple of 7? *Ans.* 10, one of them being 23, 327.

8. Required the least number which is divisible by 5 and 7, but leaves 1 when divided by 6. *Ans.* 175.

9. Divide 100 into three such parts that the first may be divisible by 13, the second by 15, and the third by 27.

Ans. 13, 60, 27.

10. A man buys oxen and horses for $1000, he gives $19 for each ox, and $29 for each horse. How many did he buy?

Ans. 45 oxen, and 5 horses; or 16 oxen, and 24 horses.

11. Divide the fraction $\frac{118}{99}$ into two others whose denominators shall be 9 and 11. *Ans.* $\frac{5}{9}$ and $\frac{7}{11}$.

SUGGESTION.—Let x and y represent the numerators of the fractions, then $\frac{x}{9}+\frac{y}{11}=\frac{118}{99}$, or $11x+9y=118$, whence $x=9p-4$, and $y=18-11p$.

12. Find three fractions, whose sum is $\frac{661}{385}$, and whose denominators are 5, 7, and 11. *Ans.* $\frac{3}{5}$, $\frac{4}{7}$, $\frac{6}{11}$.

13. Find the least number which, being divided by 28, 19, and 15, shall leave respectively the remainders 19, 15, and 11.

Ans. 7691.

14. Divide 200 into two such parts, that if one of them be divided by 6, and the other by 11, the respective remainders may be 5 and 4.

SUGGESTION.— Let x and y denote the quotients, then the parts will be $6x+5$, and $11y+4$; and $6x+11y=191$.

Ans. 185 and 15, 119 and 81, or 53 and 147.

15. In what year of the Christian era, was the solar cycle 8, the lunar cycle 10, and the Roman Indiction 10? *Ans.* 1567.

16. A shepherd has a flock of sheep less than 200; when he counts them by fours, sixes, or nines, he has 3 over each time; when he counts them by sevens or thirteens he has 1 over, and when he reckons them by elevens the remainder is 7. How many sheep has he? *Ans.* 183.

17. A person wishes to purchase 20 animals for 20£. (400 shillings); viz.. sheep at 31s., pigs at 11s., and rabbits at 1s. each; in how many ways can he do it? *Ans.* Three ways; one of which is 12 sheep, 2 pigs, and 6 rabbits.

18. A wheel in 36 revolutions passes over 29 yards; and in x of these revolutions it describes z yards $+y$ feet $+5$ inches; required the values of x, y, and z. *Ans.* $x=13$, $y=1$, $z=10$.

DIOPHANTINE ANALYSIS.

ART. 14. The object of the Diophantine Analysis is, to render algebraic expressions containing one or more unknown quantities, exact powers, such as squares or cubes; or, what amounts to the same, to find such values of a quantity as shall render a radical expression depending on it rational.

Ex. Let it be required to find such values of x as shall render $4x+5$ a square; that is, so that $\sqrt{4x+5}$ can be exactly determined. If we assume $4x+5=m^2$, we find $x=\frac{m^2-5}{4}$, where m may be any number whose square is greater than 5. If $m=3$ $x=1$; if $m=4$, $x=2\frac{3}{4}$.

REMARK.— The Diophantine is properly a branch of Indeterminate Analysis; it derives its name from Diophantus of Alexandria, in Egypt who lived about A. D. 350. In its full extent it is a comprehensive subject, and has occupied the attention of some of the greatest mathematicians. The following is designed to present merely the elementary principles of the subject. Those who desire a thorough knowledge of it are referred to "Euler's Algebra, with Lagrange's additions," "Barlow's Theory of Numbers," and "Legendre Theorie des Nombres."

ART. 15 PROBLEM I.—*To find such values of x as will render rational the expression*

$$\sqrt{ax^2+bx+c}.$$

The solution of this problem assumes different forms, depending on the values of a, b, and c.

ART. 16. CASE I.—*When $a=0$, or when the expression becomes*

$$\sqrt{bx+c}.$$

Let $\sqrt{bx+c}=p$, where p may be *any* number whatever

then $bx+c=p^2$, and $x=\frac{p^2-c}{b}$.

EXAMPLES.

1. Five times a certain number, diminished by 4, makes a square, required the number.

Here $b=5$, $c=-4$, and $x=\frac{p^2+4}{5}$, where p may be *any* number whatever.

If $p=6$, then $x=8$; if $p=1$, then $x=1$; if $p=2$ then $x=1.6$, and so on.

2. Find such values of x as will render the following expressions square numbers, and verify the result.

$$5x+3,\ 5x-3,\ 10-3x,\ 3x+\tfrac{1}{4}.$$

ART. 17. CASE II.—*When $c=0$, or when the expression becomes*

$$\sqrt{ax^2+bx}.$$

Let $\sqrt{ax^2+bx}=px$,

then $ax^2+bx=p^2x^2$, and $x=\frac{b}{p^2-a}$, where p may be any number whatever.

EXAMPLES.

1. Find such a value of x as will render $5x^2+8x$ a square.

Here $a=5$, $b=8$, and $x=\frac{8}{p^2-5}$.

If $p=3$, $x=2$, if $p=2$ $x=-8$.

2. Divide the number a into two parts, such that their product shall be a square.

Let $x=$ one part, then $a-x=$ the other, and their product is $ax-x^2$, which it is required to make a square.

Let $ax-x^2=p^2x^2$, whence $x=\frac{a}{p^2+1}$, p being any number whatever.

If $a=10$ let $p=2$, then $x=2$, and $a-x=8$.

3. Find such a value of x as shall render $7x^2-15x$ a square.

$$x=\frac{15}{7-p^2}\text{; if } p=2, x=5.$$

4. Required a number such, that if its half be added to double its square, the result shall be a square.

$$x=\frac{1}{2p^2-4},\ p \textit{ being any number.}$$

ART. 18. CASE III.— *When a is a square, or when the expression is of the form* $\sqrt{a^2x^2+bx+c}$.

Let $\sqrt{a^2x^2+bx+c}=ax+p$,

then $a^2x^2+bx+c=a^2x^2+2apx+p^2$,

whence $x=\frac{c-p^2}{2ap-b}$, or $\frac{p^2-c}{b-2ap}$.

EXAMPLES.

1. Find such a value of x as will render $4x^2+3x-7$ a square.

Here $a=2$, $b=3$, and $c=-7$, whence $x=\frac{p^2+7}{3-4p}$. Let $p=\frac{1}{2}$, then $x=7\frac{1}{4}$.

2. Find a number, such that if it be increased by 2 and 5 separately, the product of the sums shall be a square.

Let $x=$ the number, then it is required to make $(x+2)(x+5)$ a square. If $p=4$, $x=\frac{2}{5}$.

3. Find a number, such that twice the number increased by 1, multiplied by eight times the number diminished by 2, shall be a square.

Let $x=$ the number, then it is required to make $(2x+1)(8x-2)$ a square. If $p=\frac{1}{4}$, $x=1\frac{1}{32}$.

ART. 19. CASE IV.— *When c is a square, or when the expression is of the form* $\sqrt{ax^2+bx+c^2}$.

$$\text{Let } \sqrt{ax^2+bx+c^2}=px+c,$$

$$\text{then } ax^2+bx+c^2=p^2x^2+2pcx+c^2.$$

$$\text{whence } x=\frac{2pc-b}{a-p^2}.$$

EXAMPLES.

1. Find such a value of x as shall render $3x^2+5x+9$ a square. If $p=1$, $x=\frac{1}{2}$.

2. Divide the number 16 into two parts, such that the sum of their squares shall be a square.

Let x and $16-x$ represent the parts, then it is required to make $2x^2-32x+256$ a square. If $p=3$, then the parts are $9\frac{1}{7}$, and $6\frac{6}{7}$.

Similarly, we may find two numbers whose difference shall be equal to a given number d, and the sum of whose squares shall be a square.

ART. 20. CASE V.— *When neither a nor c are squares, but when* b^2-4ac *is a square.*

Let r and r' be the roots of the equation

$$x^2+\frac{b}{a}x+\frac{c}{a}=0;$$

$$\therefore ax^2+bx+c=a(x-r)(x-r').$$

$$\text{Let } \sqrt{ax^2+bx+c}=p(x-r'),$$

$$\text{then } ax^2+bx+c=p^2(x-r')^2;$$

$$\therefore a(x-r)(x-r')=p^2(x-r')^2,$$

$$a(x-r)=p^2(x-r');$$

$$\text{whence } x=\frac{ar-p^2r'}{a-p^2}.$$

Now the values of r and r' are

$$r=\frac{-b+\sqrt{b^2-4ac}}{2a}, \text{ and } r'=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

which will be rational when b^2-4ac is a square.

Let $b^2-4ac=d^2$, then

$$r=\frac{d-b}{2a}, \text{ and } r'=\frac{-d-b}{2a}.$$

$\therefore$ by substitution $x=\dfrac{ad-ab+p^2(b+d)}{2a(a-p^2)}$.

EXAMPLES.

1. Find such a value of x as will render the expression $6x^2+13x+6$ a square.

Here $b^2-4ac=169-144=25$, and $d=5$.

$$\therefore\ x=\frac{30-78+18p^2}{72-12p^2}=\frac{3p^2-8}{12-2p^2}\text{; let } p=2\text{, then } x=1.$$

If $p=2\frac{1}{3}$, $x=7\frac{1}{2}$.

2. Find such a value of x as will render $2x^2+10x+12$ a square.

Ans. $x=\dfrac{3p^2-4}{2-p^2}$; if $p=\frac{4}{3}$, $x=6$; if $p=\frac{5}{4}$, $x=\frac{11}{7}$.

3. Find such a value of x as will render $3x^2-8x+5$ a square.

Ans. $x=\dfrac{5-p^2}{3-p^2}$; if $p=1$, $x=2$.

ART. 21. CASE VI. *When the proposed expression can be separated into two parts, one of which is a square, and the other the product of two factors.*

If none of the preceding methods be applicable, still the solution can be effected, if the proposed expression is equal to a square increased or diminished by the product of two factors. The difficulty, however, consists in decomposing the expression, which can only be done by trial.

If $ax^2+bx+c=(dx+e)^2+(fx+g)(hx+k)$, let the latter $=\{dx+e+p(fx+g)\}^2$. Squaring this, omitting equal quantities on each side, and reducing

$$x=\frac{p(2e+pg)-k}{h-p(2d+pf)}.$$

EXAMPLES.

1. What value of x will render $5x^2-1$ a square?

By trial we find $5x^2-1=(2x)^2+(x-1)(x+1)$.

Comparing this with the formula, we have $d=2, e=0, f=1$ $g=-1$, $h=1$, and $k=1$; whence $x=\frac{p^2+1}{p^2+4p-1}$. If $p=1, x=\frac{1}{2}$; if $p=\frac{1}{2}, x=1$.

2. What value of x will render $2x^2+8x+7$ a square?

Here $2x^2+8x+7=(x+2)^2+(x+1)(x+3)$.

If $p=3, x=3$.

ART. 22. When all the preceding methods fail, we may often find, by trial, such a value r of x, as shall render ax^2+bx+c a square. Having done this, substitute $y+r$ for x, and the resulting equation will be $a(y+r)^2+b(y+r)+c=ay^2+2ary+by+ar^2+br+c$; but by hypothesis ar^2+br+c is a square; calling this n^2, the expression becomes $ay^2+2ary+n^2$, which can now be rendered a square by Case IV, Art. 19.

EXAMPLES.

1. Find such values of x as will render $6x^2-10x-3$ a square.

By trial, we find $x=2$ renders the expression a square. Let $x=y+2$; then by substitution, and reduction, the expression becomes $6y^2+14y+1$. Let this $=(py+1)^2$, then $y=\frac{2p-14}{6-p^2}$, and since $x=y+2$, we have $x=\frac{2p^2-2p+2}{p^2-6}$, from which, by giving various values to p, we may find as many values of x as we please.

2. Find a general expression for the value of x that will render $10+8x-2x^2$ a square, which is a square when $x=1$.

Ans. $x=\frac{8p+4}{p^2+2}+1$.

PROBLEM II. *To find such values of x as will render the expression ax^3+bx^2+cx+d a square.*

There are but two cases in which this problem admits of a direct solution: 1st, when the last two terms are wanting; or 2d, when the last term is a square.

ART. 23. CASE I. *When the expression is of the form bx^3+bx^2*

Let $ax^3+bx^2=(px)^2=p^2x^2$,

then $x=\frac{p^2-b}{a}$.

EXAMPLES.

1. Find x such that $2x^3+3x^2$ shall be a square.

If $p=3$, $x=3$; if $p=5$, $x=11$.

2. Find a number, such that 5 times its cube, increased by 10 times its square, shall be a square. *Ans.* $x=3$.

3 Find a number, such that 3 times its cube, diminished by 10 times its square, shall be a square. *Ans.* $x=5$.

ART. 24. CASE II. *When the expression is of the form* $ax^3+bx^2+cx+d^2$.

Let $ax^3+bx^2+cx+d^2=\left(\frac{c}{2d}x+d\right)^2=\frac{c^2}{4d^2}x^2+cx+d^2$;

$$\text{whence } x=\frac{c^2-4bd^2}{4ad^2}.$$

1. Find such a value of x as shall render x^3-x^2+2x+1 a square. Here $\frac{c}{2d}=1$. *Ans.* $x=2$.

2. What value of x will render $3x^3-5x^2+6x+6$ a square? *Ans.* $x=\frac{29}{12}$.

3. What value of x will render $2x^3-5x^2+12x+4$ a square? *Ans.* $x=7$.

ART. 25. If we know one value r of x, that will render ax^3+bx^2+cx+d a square, we may find others as follows:

Let $ar^3+br^2+cr+d=m^2$, and transform the equation $ax^3+bx^2+cx+d=0$ into another whose roots shall be $x-r$, (Algebra, Art. 406); the transformed equation will be of the form

$$ay^3+b'y^2+c'y+m^2=0.$$

We may then, by Art. 24, find a value q of y which will render this expression a square, then the general value of x will be $x=q+r$.

Ex. Find such a value of x, other than 2, as will render x^3-x^2+2x+1 a square.

By substituting $y+2$ for x, the resulting equation is

$$y^3+5y^2+10y+9.$$

By assuming this equal to $(\frac{5}{3}y+3)^2$, and reducing, we find

$$y=-\tfrac{20}{9} \quad \therefore\ x=-\tfrac{20}{9}+2=-\tfrac{2}{9}.$$

Problem III. *To find such values of x as shall render* $ax^4+bx^3+cx^2+dx+e$ *a square.*

Art. 26. Case I. *When the first term only is a square, that is, to make* $a^2x^4+bx^3+cx^2+dx+e$ *a square.*

Let $a^2x^4+bx^3+cx^2+dx+e=(ax+mx+n)^2=a^2x^4+2amx^3+(m^2+2an)x^2+2mnx+n^2$.

In order that the first three terms on each side shall be the same, we must make

$$\left.\begin{array}{l} b=2am \\ c=m^2+2an \end{array}\right\} \text{ whence } \left\{\begin{array}{l} m=\dfrac{b}{2a} \\ n=\dfrac{c-m^2}{2a}=\dfrac{4a^2c-b^2}{8a^3}; \end{array}\right.$$

this gives $dx+e=2mnx+n^2$, and $x=\dfrac{n^2-e}{d-2mn}$.

EXAMPLES.

1. What value of x will render x^4-3x+2 a square? *Ans.* $x=\frac{2}{3}$.

2. Required a value of x, such that the expression $4x^4+4x^3+4x^2+2x-6$ may be a square. *Ans.* $x=13\frac{1}{8}$.

Art. 27. Case II. *When the last term only is a square, that is, to make* $ax^4+bx^3+cx^2+dx+e^2$ *a square.*

Let $x=\dfrac{1}{y}$, then the expression becomes

$$\frac{a+by+cy^2+dy^3+e^2y^4}{y^4}.$$

The numerator of this expression may be rendered a square by the preceding article, and the denominator is already a square, therefore the whole will be a square.

EXAMPLES.

1. What value of x will render $2x^4-3x^3+1$ a square?

Here it is required to make y^4-3y+2 a square. $y=\frac{2}{3}$ $\therefore$ $x=\frac{3}{2}$.

When the first and last terms are both squares, the problem may be solved by either of the preceding cases.

1. What value of x will render $x^4-6x^3+4x^2-24x+16$ a square? *Ans.* $x=\frac{1}{4}$.

Art. 28. We might now proceed to consider how an expression of the form $ax^4+bx^3+cx^2+dx+e$ can be rendered a square.

The general principle is, to assume the given expression equal to such a quantity, that, after squaring, all the terms may disappear, or be made to do so, except those containing two consecutive powers of x; as the value of this quantity can then be obtained in a rational form. The terms to be destroyed may be at the beginning of the given expression, or at its end, or both, according to its nature.

Ex. 1. What value of x will render $4x^4+12x^3-3x^2-2x+1$ a square.

Assume this equal to $(2x^2+px+q)^2$, then squaring and reducing, we have $12x^3-3x^2-2x+1=4px^3+(p^2+4q)x^2+2pqx+q^2$.

Equating the coëfficients 12 and $4p$, and also -3 and p^2+4q, we get $p=3$, and $q=-3$; this reduces the equation to $-2x+1=2pqx+q^2=-18x+9$; whence $x=\frac{1}{2}$.

When we know one value of the unknown quantity that satisfies the conditions other values may be found as in Art. 26.

Ex. What other value of x, besides 1, will render $3x^4-2$ a square?

By substituting $y+1$ for x, we get $3y^4+12y^3+18y^2+12y+1$.

Since the last term is a square, assume this equal to $(py^2+qy+1)^2$, then the last three terms in each member will disappear by taking $p=-9$, and $q=6$; whence $y=\frac{20}{13}$, and therefore $x=\frac{33}{13}$.

By assuming $x=y+\frac{33}{13}$, and performing a similar process, we can find another value of x, and so on.

Art. 29. Quantities of the form ax^3+bx^2+cx+d, can be rendered cubes on principles exactly similar to those that have been employed in rendering quantities squares. Thus, if a be a cube, we may destroy the first and second terms; if d be a cube, the third and fourth terms can be destroyed: if a and d be both cubes, we can destroy the first and last terms. When neither a nor d is a cube, if we can find a value r of x, which being substituted for x, will render the expression a cube, we may substitute $y+r$ for x, and then obtain an expression which can be rendered a cube by the principles just explained. As an example let it be required to find a value of x which will make $2x^3+3$ a cube.

Here we see that $x=-1$ satisfies the conditions; to find another value substitute $y-1$ for x, and the expression becomes $2y^3-6y^2+6y+1$. Assume this equal to $(py+1)^3$, and then by taking $p=2$, to make the last two terms of each member disappear, we get $y=-3$, therefore $x=-4$. By substituting $y-4$ for x, we might obtain another value, and so on.

EXAMPLES.

1. Find a value of x that will render $3x^3+2x+1$ a cube.
Ans. $x=\frac{36}{73}$.

2. Find a value of x, besides $x=1$, that will render $2x^3-4x+6x+4$ a cube.
Ans. $x=\frac{17}{53}$.

3. Find a value of x, besides $x=-1$, that will render x^2+x+1 a cube.
Ans. $x=-19$.

DOUBLE AND TRIPLE EQUALITIES.

ART. **30.** *Double*, *triple*, or *higher equalities*, are problems in which two, three, or more functions of a quantity are to be made squares or cubes, for the *same value* of x. Thus, if it be required to find a value of x that shall render both the expressions, $ax+b$, and $cx+d$ squares, the problem presents a double equality.

In the following solutions, for the sake of brevity, the symbol $\square$ is used to represent a square number.

PRINCIPLE — It is sometimes convenient to use the following principle:

If a square be multiplied by a square, the product will be a square, or if a square be divided by a square, the quotient will be a square.

ART. **31.** CASE I.— *To solve the double equality*

$$ax+b=\square,$$

$$cx+d=\square.$$

Let $ax+b=p^2$, and $cx+d=q^2$, then equating the two values of x, and reducing, we find

$$c^2p^2=caq^2-cad+c^2b.$$

Since the left member is a $\square$, q must have such a value as to render the right member a $\square$, which may be ascertained by some of the preceding methods.

EX. What value of x will render $x-1$ and $2x-1$ both squares?
Ans. $x=5$, or $x=\frac{65}{49}$

ART. 32. CASE II.— *To solve the double equality.*

$$ax^2+bx=\square,$$
$$cx^2+dx=\square.$$

Let $x=\frac{1}{y}$ then we have

$$\frac{1}{y^2}(a+by)=\square,$$
$$\frac{1}{y^2}(c+dy)=\square.$$

Hence, by the principle, Art. 30, it is only necessary to render $a+by$, and $c+dy$ both squares, which belongs to the preceding problem.

ART. 33. CASE III.— *To solve the double equality*

$$ax^2+bx+e=\square,$$
$$dx^2+ex+f=\square.$$

Here we must solve the equality $ax^2+bx+c=\square$, by methods already explained, and then substitute the value of x so found in the equality $dx^2+ex+f=\square$, which will rise to the fourth degree, and which must then be solved by methods explained in Art. 28.

ART. 34. CASE IV.— *To solve the triple equality*

$$ax+by=\square,$$
$$cx+dy=\square,$$
$$ex+fy=\square.$$

Put $ax+by=t^2$, $cx+dy=u^2$, and $ex+fy=s^2$.

By eliminating y from the first two equations, and x from the same equations, we find $x=\frac{dt^2-bu^2}{ad-bc}$, and $y=\frac{au^2-ct^2}{ad-bc}$; substituting these for x and y in the third equation; putting $u=tz$, and dividing the expression by t^2, we have

$$\frac{(af-bc)z^2-(cf-de)}{ad-bc}=.\square$$

When it is possible the value of z may be found by Problem 1, Articles 15 to 22.

Having the value of z, we may assume t of any convenient value, this will give the value of u; then by substitution, the values of x and y are easily obtained.

The preceding are the most general methods hitherto discovered. In the resolution of most problems, however, much will depend on the judgment and skill of the operator, and the most important and difficult problems are solved by methods for which no special rules can be given.

MISCELLANEOUS EXERCISES.

1. To divide a given square number, a^2, into two squares.

Let $x^2=$ one part, then $a^2-x^2=$ the other. Assume $a^2-x^2=(a-vx)^2$, whence $x=\frac{2av}{v^2+1}$.

$$\therefore \text{ the parts are } \frac{a^2(v^2-1)^2}{(v^2+1)^2}, \text{ and } \frac{4a^2v^2}{(v^2+1)^2}.$$

Suppose $a^2=100$, then if $v=2$, the parts are 36 and 64, if $v=4$, the parts are $\frac{22500}{289}$, and $\frac{6400}{289}$, and so on.

By means of this formula we can divide a given square into any assigned number of squares, by first dividing it into two squares, and then subdividing one or both of these into others.

The solution of this problem gives the following equation;

$$a^2=\frac{a^2(v^2-1)^2}{(v^2+1)^2}+\frac{4a^2v^2}{(v^2+1)^2}.$$

Dividing both members by a^2, and multiplying by $(v^2+1)^2$ we have
$$(v^2+1)^2=(v^2-1)^2+4v^2;$$

Substituting $\frac{p}{q}$ for v, and multiplying both members by q^4, we get
$$(p^2+q^2)^2=(p^2-q^2)^2+4p^2q^2.$$

Hence, the square of p^2+q^2 being equal to the sum of the squares of p^2-q^2 and $2pq$, it follows (Legendre IV, 11,) that if p^2+q^2 be the hypothenuse of a right-angled plane triangle, p^2-q^2 and $2pq$ will be its legs; this gives the following useful

RULE.— *To find the sides of a right-angled triangle in whole numbers, take two unequal whole numbers; then the sum of their squares, the difference of their squares, and twice their product, will be the three sides.*

Thus, by taking 1 and 2, we find the sides to be 5, 3, and 4; if we take 1 and 3 the sides will be 10, 8, and 6.

2. To divide a number which is the sum of two known squares, a^2 and b^2, into two other squares.

Let x^2 be one of the parts, then $a^2+b^2-x^2=\square$. But one value of x is b or a; therefore, substitute $y+b$ (Art. 22,) for x, and we get $a^2-y^2-2by=\square$; assume this equal to $(a-vy)^2$, and we find $y=\dfrac{2av-yb}{v^2+1}$, $\therefore\ x=y+b=\dfrac{b(v^2-1)+2av}{v^2+1}$.

Example. Let the given number be $185=4^2+13^2$. Here $a=4$, and $b=13$; let $v=2$, then $x=11$, and $x^2=121$, and $185-121=64=8^2$. If $v=4$ the parts are $(\frac{227}{17})^2$, and $(\frac{44}{17})^2$.

3. To find three square numbers in arithmetical progression.

Assume $(x-y)^2$, x^2+y^2, and $(x+y)^2$ for the three numbers, whose common difference is $2xy$, and of which the first and third are already squares. It only remains then to make x^2+y^2 a square, which may be done in the manner explained in the latter part of the solution to Ex. 1. Thus, let the two unequal numbers be 2 and 1, then x, (the diff. of their squares,) $=3$, and y, (twice their product,) $=4$. Hence $(x-y)^2$, x^2+y^2, and $(x+y)^2$ are 1, 25, and 49.

4. To find any assigned number (n) of squares whose sum shall be a square.

By assuming as the required squares a^2, b^2, c^2. . . and x^2, where a^2, b^2, &c., are numbers assumed at pleasure, it only remains to find such a value of x as shall make $a^2+b^2+c^2$. . $+x^2$ a square, which may be done by assuming it equal to $(x+p)^2$, and resolving the equation so found for x.

5. Find two whole numbers, such that their difference shall be a square, and the sum of their squares a cube.

Assume $4x^2$ and $3x^2$ for the numbers; then $4x^2-3x^2=x^2=\square$, and it remains to make $(4x^2)^2+(3x^2)^2=25x^4$ a cube.

Let $25x^4=a^3x^3$, then $x=\frac{1}{25}a^3$. Let $a=5$, then $x=5$, and the numbers are 100 and 75.

6. To find three numbers in arithmetical progression, such that the sum of every two of them may be a square.

Let $x-y$, x, and $x+y$ represent the numbers, then $2x$, $2x-y$, and $2x+y$ must be squares.

Assume $2x=r^2+s^2$, and $y=2rs$, then the second and third will be squares, and it only remains to make r^2+s^2 a square, which will be accomplished by making $r=m^2-n^2$, and $s=2mn$. This gives $2x=(m^2+n^2)^2$, and the three numbers are $\frac{1}{2}(m^2+n^2)^2-4mn(m^2-n^2)$, $\frac{1}{2}(m^2+n^2)^2$, and $\frac{1}{2}(m^2+n^2)^2+4mn(m^2-n^2)$. If $m=9$ and $n=1$, the numbers are 482, 3362, and 6242.

This question may also be readily solved by assuming the three numbers $2x^2-y$, $2x^2$, and $2x^2+y$, and then putting $y=4x-1$

7. To divide universally any given whole number, N, into as many different square numbers as it contains units.

Let $ax-1$, $bx-1$, $cx-1$, &c., continued to N terms, represent a series of roots, the sum of whose squares is to be N. Let each of these be squared separately, and put the sum of all the coëfficients of x^2, that is, $a^2+b^2+c^2+$, &c., $=m$, and those of x, that is, twice the sum of a, b, c, &c., $=n$, we shall then have $mx^2-nx+N=N$; from which $x=\frac{n}{m}$. To apply this to a particular question let it be required to divide the number 4 into 4 square numbers.

Let a, b, c, and $d=2$, 3, 4, and 5, then $m=a^2+b^2+c^2+d^2=4+9+16+25=54$; $n=2a+2b+2c+2d=4+6+8+10=28$ $\therefore \frac{n}{m}=\frac{28}{54}=\frac{14}{27}$, and $ax-1$, $bx-1$, $cx-1$, and $dx-1=\frac{1}{27}$, $\frac{15}{27}$, $\frac{29}{27}$, and $\frac{43}{27}$, and the numbers are $(\frac{1}{27})^2$, $(\frac{15}{27})^2$, $(\frac{29}{27})^2$, and $(\frac{43}{27})^2$, or $\frac{1}{729}$, $\frac{225}{729}$, $\frac{841}{729}$, and $\frac{1849}{729}$.

8. Find three cube numbers, whose sum shall be a cube.

Put x^3, y^3, and z^3 for the three cubes, and let their sum

$$=(x+z)^3=x^3+3x^2z+3xz^2+z^3,$$

$$\therefore y^3=3x^2z+3xz^2.$$

Put $x=pz$, then $y^3=3p^2z^3+3pz^3=z^3(3p^2+3p)$.

It is now required to make $3p^2+3p$ a cube, which it is when $p=\frac{1}{8}$; consequently if we make $z=8$. $pz=1$, and $y=6$; $\therefore$ the three cubes are 1^3, 6^3, and 8^3, whose sum is 9^3. By making $z=$ any multiple of 8, we may obtain as many integral solutions as we please.

The learner should observe that it is generally important to assume the numbers so as to satisfy as many of the conditions as possible.

9. Find two numbers, such that their sum and difference shall be squares *Ans.* v^2+1, and $2v$, or 5 and 4, &c.

10. Find two numbers, such that if each be added to the square of the other, the sum shall be a square. *Ans.* $\frac{4(v^2+1)}{8v+1}$, and $\frac{v^2-8v}{v+1}$, which are found by assuming $4x$ and $x-1$ for the numbers

11. Find two numbers, such that the difference of their cubes may be a square number. *Ans.* $\frac{2v+3}{v^2-3}$, and $\frac{v^2+2v}{v^2-3}$.

12. Find a number, such that the sum of its square and cube may be a square. *Ans.* v^2-1, or 3, 8, &c.

13. Find two numbers, such that if to each of them, and to their sum and difference 1 be added, each of the four sums may be a square. *Ans.* 168, and 120.

Let x^2+2x and x^2-2x represent the numbers.

14. Render $2x^2-2$ a square. *Ans.* $x=\frac{v^2+2}{v^2-2}$.

First assume $x=y+1$.

15. Find two numbers, such that the difference of their squares may be a cube, and the difference of their cubes a square.

Ans. $10v^6$, and $6v^6$

Assume the numbers equal to x^3+2a^6, and x^3-2a^6.

16. Find a number, such that if 1 be added to its double and triple, each of the results may be a square.

Let $2x^2+2x$ represent the number, then $x=\frac{2v+6}{v^2-6}$.

Ans. 40, 3960, &c.

17. Required three numbers, such that the sum of all three, and the sum of every two of them may be a square number.

Assume $4x$, x^2-4x, and $2x+1$ for the three numbers.

Ans. $\frac{2}{3}(v^2-1)$, $\frac{1}{36}(v^2-1)^2-\frac{2}{3}(v^2-1)$, and $\frac{1}{3}(v^2+2)$.

18. Find two numbers, such that their difference may be equal to the difference of their squares, and that the sum of their squares may be a square. *Ans.* $\frac{2v-2}{v^2-2}$, and $\frac{v^2-2v}{v^2-2}$.

19. Find two square numbers, whose sum shall be equal to their product. *Ans.* $\frac{(v^2+1)^2}{(v^2-1)^2}$, and $\frac{(v^2+1)^2}{4v^2}$, or $\frac{25}{9}$, and $\frac{25}{16}$, &c.

20. To divide a number which is the sum of three square numbers in arithmetical progression, into three other squares which shall also be in arithmetical progression.

Ans. The numbers will be found by dividing one-third of the given number into two squares, a^2 and b^2, by Example 1, and taking $(a-b)^2$, a^2+b^2, and $(a+b)^2$ as the required numbers.

21. To find three whole numbers in arithmetical progression whose common difference shall be a cube, the sum of any two diminished by the third a square, and the sum of the roots of the required squares a square.

Ans. 2698071376114832,
5188598800220160 0,
7679126224325836 8.

PROPERTIES OF NUMBERS.

ART. **35.** DEFINITIONS. *Even* numbers are those which are divisible by 2.

Odd numbers are those which, when divided by 2, leave a remainder 1.

An even number is represented by the formula $2n$, and an odd number by the formula $2n+1$.

A *prime* number has no divisor except itself and unity.

Numbers are prime to each other when they have no common divisor, except unity.

ART. **36**. If $\frac{M}{N}$ be a fraction, and if d be the greatest common divisor of M and N, so that $M=ad$, and $N=bd$, then $\frac{M}{N}=\frac{a}{b}$; and $\frac{a}{b}$ is the fraction $\frac{M}{N}$ in its lowest terms, and a is prime to b.

There can be no other fraction $\frac{p}{q}$ when p is prime to q, which shall be equal to $\frac{M}{N}$; for if so, M and N would have two greatest common measures, which is absurd.

ART. **37.** I. *Every number* N *may be expressed by the formula*

$$N=qn+r.$$

For if the number N be divided by q, and if n be the quotient and r the remainder, then by the principles of division,

$$N=qn+r;$$

q is called the *modulus*, and by giving different values to q, different forms of numbers may be obtained. It is evident that r can not exceed $q-1$.

ART. 38. II. *Every number is of one of the forms, $3n$, or $3n\pm1$.*

Comparing this with the general formula $N=qn+r$, we have

$$q=3,\ r=0,\ 1,\ \text{or}\ 2.$$

$$\therefore\ N=3n,\ \text{or}\ 3n+1,\ \text{or}\ 3n+2;$$

$$\text{but}\ 3n+2=3n+3-1=3(n+1)-1=3n'-1.$$

$\therefore$ every number is one of the forms $3n$, or $3n\pm1$.

ART. 39. III. *Every square number is of one of the forms $4n$, or $4n+1$.*

Every number is either $2n$, or $2n+1$.

If $N=2n$; $N^2=4n^2=4n'$, if $n^2=n'$.

If $N=2n+1$; $N^2=4n^2+4n+1=4n(n+1)+1=4n'+1$, if $n(n+1)=n'$.

Hence, N^2 is of the form $4n$, or $4n+1$; that is, every square number is either divisible by 4, or, when divided by 4, leaves unity for its remainder.

ART. 40. IV. *The difference between the squares of any two odd numbers is divisible by 8.*

Let M and N be the numbers, M being $>N$.

Also, let $M=2m+1$, and $N=2n+1$,

$$\therefore\ M^2-N^2=4(m^2-n^2)+4(m-n)$$

$$=4(m+n)(m-n)+4(m-n)=4(m-n)(m+n+1),$$

which is evidently divisible by 4; and since whatever values be given to m and n, either $m-n$ or $m+n+1$ is an even number;

$$\therefore\ M^2-N^2\ \text{is divisible by}\ 4\times2\ \text{or}\ 8.$$

ART. 41. V. *The product of three successive numbers is divisible by 2×3, or $\dfrac{n(n+1)(n+2)}{2\times3}$ is an integer.*

For one of the first two factors must be an even number, and one of the three must be of the form $3m$, or is divisible by 3; $\therefore$ since 2 and 3 are prime factors, $n(n+1)(n+2)$ must be divisible by 2 and 3.

Similarly, it may be proved that the product of four successive numbers is divisible by $2\times3\times4$; that the product of five successive numbers is divisible by $2\times3\times4\times5$, and so on.

ART. 42. VI. *The difference between a number and its cube is divisible by* 6.

For $n^3-n=n(n^2-1)=(n-1)n(n+1)$, which, being the product of 3 successive numbers, is divisible by 2×3, or 6.

Hence $\frac{n^3}{6}$ and $\frac{n}{6}$ leave the same remainder.

ART. 43. VII. *If n be a whole number,* $n(n^2-1)(n^2-4)$ *is divisible by* 120.

For $n(n^2-1)(n^2-4)=n(n-1)(n+1)(n-2)(n+2)$
$=(n-2)(n-1)n(n+1)(n+2)$,
which, being the product of 5 consecutive numbers, is divisible by $2\times3\times4\times5$, or 120, (Art. 41).

ART. 44. VIII. *Every square number is either* $5n$, *or* $5n\pm1$.

Every number is of one of the forms $5n, 5n+1, 5n+2, 5n+3, 5n+4$; all of which are included in the forms $5n$, $5n\pm1$, $5n\pm2$, since $5n+3=5(n+1)-2=5n'-2$, and $5n+4=5(n+1)-1=5n'-1$.

But $(5n)^2=5(5n^2)=5n'$, which is of the form $5n$;

$(5n\pm1)^2=25n^2\pm10n+1=5(5n^2\pm2n)+1$, which is of the form $5n+1$;

$(5n\pm2)^2=25n^2\pm20n+4=5(5n^2\pm4n+1)-1$, which is of the form $5n-1$.

$\therefore$ every square is of one of the forms $5n$, or $5n\pm1$.

ART. 45. IX. *Every cube number is either* $7n$, *or* $7n\pm1$.

For every number is of one of the forms $7n, 7n+1, 7n+2, 7n+3, 7n+4, 7n+5, 7n+6$. But $7n+4=7n+7-3$, $7n+5=7n+7-2$, $7n+6=7n+7-1$, hence every number is either $7n, 7n\pm1, 7n\pm2$, or $7n\pm3$;

$\therefore$ N^3 is $(7n)^3$, $(7n\pm1)^3$, $(7n\pm2)^3$, or $(7n\pm3)^3$, of which the first two are of the required forms.

But $(7n\pm2)^3=(7n)^3\pm6(7n)^2+12(7n)\pm8=7m\pm8$
$=7(m+1)+1=7n'+1$;

$(7n\pm3)^3=(7n)^3\pm9(7n)^2+27(7n)\pm27=7m\pm27$
$=7(m\pm4)-1=7n'-1$;

$\therefore$ every cube number is either $7n$, or $7n\pm1$.

ART. 46. X. *Every prime number greater than* 3 *is of the form* $6n\pm1$.

For every number is of one of the forms $6n, 6n+1, 6n+2$

$6n+3, 6n+4, 6n+5$, of which the first, third, fourth, and fifth are divisible by 2; if therefore the number be a prime number it must be one of the forms $6n+1$, or $6n+5$; but $6n+5=6n+6-1=6n'-1$; $\therefore$ every prime number greater than 3, is $6n\pm1$

Cor. Hence, if any prime number is increased and diminished by unity, either the sum or the difference is divisible by 6. Thus,

$$29+1=30;\ 37-1=36.$$

ART. 47. XI. *If m be a prime number, greater than* 3, m^2-1 *is divisible by* 24.

Let $m=6n\pm1$; $\therefore$ $m^2=36n^2\pm12n+1$;

$$\therefore\ m^2-1=12n(3n\pm1),$$

and since either n or $3n\pm1$ must be an even number, and therefore divisible by 2, therefore $12n(3n\pm1)$ is divisible by 12×2 or 24; $\therefore$ m^2-1 is divisible by 24.

ART. 48. XII. *Every number is a prime which is not divisible by a number less than its square root.*

For every number N which is not a prime is composed of two factors, as a and b, so that $N=ab$.

Now if $a=b$, N is a square number, and $a=\sqrt{N}$; but if a do not $=b$, and $a>b$, then $a>\sqrt{N}$ and $b<\sqrt{N}$, that is, b a divisor of N is less than $<\sqrt{N}$; and this is obviously true for every number not a prime; $\therefore$ if a number is not divisible by a number less than its square root, it must be a prime.

Ex. 97 is a prime number, since it is not divisible by any number less than $\sqrt{97}$ or 10.

The following formulas contain a great number of primes by making $x=0, 1, 2, 3$, &c.

The first 40 terms of x^2+x+41 are primes,
the first 29 terms of $2x^2+29$ are primes, and
the first 31 terms of 2^x+1 are primes.

ART. 49. XIII. *If* N *be any number having prime factors, a, b, c, &c., and a is taken as a factor m times, b as a factor n times, and c as a factor r times, then* $N=a^m\,.\,b^n\,.\,c^r$.

For $N=a$ taken m time $\times b$ taken n times $\times c$ taken r times,

$$=a^m\times b^n\times c^r.$$

Ex. Let $N=360=2\times2\times2\times3\times3\times5=2^3\times3^2\times5'$.

ART. 50. XIV. *To find the number of divisors of a given number.*

Let N, the given number, $=a^m \cdot b^n \cdot c^r$, &c.

Then it is evident that N will be divisible by

$$1 \;,\; a \;,\; a^2 \;,\; a^3 \;,\; \&c. \;\ldots\ldots\; a^m;$$
$$1 \;,\; b \;,\; b^2 \;,\; b^3 \;,\; \&c. \;\ldots\ldots\; b^n;$$
$$1 \;,\; c \;,\; c^2 \;,\; c^3 \;,\; \&c. \;\ldots\ldots\; c^r;$$

and also by every possible combination of the products of these terms; that is, by every term of the product

$$(1+a+a^2, \&c., +a^m)(1+b+b^2, \&c., +b^n)(1+c+c^2, \&c., +c^r) \&c.$$

But the number of terms of this product, since no two of them can be the same, is

$(m+1)(n+1)(r+1)$, &c., which is the number of divisors of N.

Observe that unity and N are included in the number of divisors.

Ex. Find how many numbers are there by which 360 is divisible.

$360=2^3\times3^2\times5^1$; $\therefore$ $m=3$, $n=2$, and $r=1$;

$\therefore$ number of divisors $=4\times3\times2=24$.

ART. 51. XV. *To find a number* N *that shall have a given number of divisors.*

Let d represent the given number of divisors, and resolve it into factors, as $d=t\times u\times v$. Take $m=t-1$, $n=u-1$, $r=v-1$ &c., and let a, b, c be any prime numbers whatever, then $N=a^m \cdot b^n \cdot c^r$, &c., as is evident from the preceding proposition

Ex. Find a number that shall have thirty divisors.

First, $30=2\times3\times5$; $\therefore$ $m=2-1=1$, $n=3-2=1$

$r=5-1=4$; $\therefore$ $N=ab^2c^4$ is the required number

If $a=2$, $b=3$, $c=5$; then $2\times3^2\times5^4=11250$.

If $a=5$, $b=3$, $c=2$; then $5\times3^2\times2^4=720$.

If $a=5$, $b=2$, $c=3$; then $5\times2^2\times3^4=1620$.

If $a=3$, $b=5$, $c=2$; then $3\times5^2\times2^4=1200$.

Each of these numbers has thirty divisors, and in the same manner various other numbers might be found having the same property, by giving a, b, and c other prime values.

ART. 52. XVI. *To find the sum of the divisors of*

$$N=a^m \,.\, b^n \,.\, c^r.$$

Since every divisor of N is contained in the product, (Art. 50), $(1+a+a^2, \&c., +a^m)(1+b+b^2, \&c., +b^n)(1+c+c^2, \&c., +c^n)$, and since by the rule for summing a geometrical series, (Alg. Art. 297),

$$1+a+a^2 \ldots\ldots +a^m=\frac{a^{m+1}-1}{a-1},$$

$$1+b+b^2 \ldots\ldots +b^n=\frac{b^{n+1}-1}{b-1}, \&c.$$

∴ the sum must be $\left(\frac{a^{m+1}-1}{a-1}\right)\left(\frac{b^{n+1}-1}{b-1}\right)\left(\frac{c^{r+1}-1}{c-1}\right)$.

Ex. 1. Find the sum of all the divisors of 360.

$$360=2^3\times3^2\times5; \text{ therefore,}$$

$$\left(\frac{2^4-1}{2-1}\right)\left(\frac{3^3-1}{3-1}\right)\left(\frac{5^2-1}{5-1}\right)=15\times13\times6=1170. \ \textit{Ans.}$$

Ex. 2. Find the sum of the divisors of 28, the number itself being excluded.

Here $28=2\times2\times7=2^2\times7$; ∴ the sum of the divisors is $\left(\frac{2^3-1}{2-1}\right)\left(\frac{7^2-1}{7-1}\right)=\frac{7\times48}{6}=56$; and rejecting the number 28 itself, the required sum is

$$56-28=28.$$

A *perfect number* is one which is equal to the sum of all its divisors (not including itself). Thus 28, which is equal to $1+2+4+7+14$, the sum of its divisors, is a perfect number.

Other perfect numbers are 6, 496, and 8128; there are only eight perfect numbers known.

EXERCISES.

1. Prove that n^3 divided by 4 can not leave 2 for a remainder, n being any whole number.

2. Prove that no number can be a square which has any one of the numbers 2, 3, 7, 8 for its last digit.

3. Prove the following properties of a square number.

(1). A square number can not terminate with an odd number of cyphers.

(2). If a square number terminates with 5, it must terminate with 25.

(3). No square number can terminate with two figures the same, except they be two ciphers, or two 4's.

3. If each of the quantities a, b, n, be a whole number, show that $\{2a+(n-1)b\}\frac{n}{2}$ is always a whole number.

4. Show that x^5-5x^3+4x is divisible by 120, when x is any positive whole number.

SUGGESTION. $x^5-5x^3+4x=x^3(x^2-4)-x(x^2-4)$.

5. Prove that if any square number be divided by 12, the remainder is a square number, that is, that it is 1, 4, or 9.

6. Find the number of divisors of 1000. *Ans.* 16.

7. Find the number of divisors of 2160, and also their sum. *Ans.* 40, and 7440.

8. Prove that the product of two different prime numbers can not be a square.

SCALES OF NOTATION.

ART. **53.** *To explain the different systems of notation.*

DEF. NOTATION is the method of representing numbers by symbols; and it comprises different *scales* dependent upon the number of the symbols or figures employed.

In the common system of notation, each figure of any number increases in value in a tenfold ratio in proceeding from right to left. Thus 5432 is equal to $5000+400+30+2$

$$=5\times1000+4\times100+3\times10+2$$
$$=5\times10^3+4\times10^2+3\times10^1+2$$

The figures 5, 4, 3, 2 are called *digits*, and the number 10, according to whose powers they proceed, is called the *radix* of the scale.

It is purely conventional that 10 should be the *radix;* the choice of it has probably arisen from the circumstance of our having *ten* fingers on the two hands. There may be any number of different *scales*, each of which has its own *radix*. When the *radix* is 2, the scale is called *Binary*; when 3, *Ternary*; when 4, *Quaternary*; when 5, *Quinary*; when 6, *Senary*; when 7, *Septenary*; when 8, *Octary*; when 9, *Nonary*; when 10, *Denary*; when 11, *Undenary*; when 12, *Duodenary*; and so on.

If 5432 represents a number in the Senary system, whose scale is 6, it may be represented thus,

$5\times6^3+4\times6^2+3\times6+2$; or, inverting the order,

$2+3\times6+4\times6^2+5\times6^3$.

And generally, if the digits of a number be a_0, a_1, a_2, a_3, &c., reckoning from *right to left*, and the radix be r, the number will be represented by

$$a_0+a_1r+a_2r^2+a_3r^3+a_4r^4+, \text{\&c.}$$

Or, if there be n digits, by reversing the order of the terms, the number will be expressed by

$$a_{n-1}r^{n-1}+a_{n-2}r^{n-2}+a_{n-3}r^{n-3}+\ .\ .\ .\ .\ +a_1r+a_0.$$

In any scale of Notation every digit is necessarily less than r, and the number of the digits, including 0, is equal to r. Also, in any number, the highest power of r is less by 1 than the number of digits.

Cor. Hence the digits including the cipher, in the

Binary scale are 1, 0.
Ternary " " 1, 2, 0.
Quaternary " 1, 2, 3, 0.
Quinary " 1, 2, 3, 4, 0. And so on.

In the Duodenary scale it will be necessary to add two characters to represent *ten* and *eleven*; we, therefore, for ten put t for eleven e.

∴ Duodenary digits are 1, 2, 3, 4, 5, 6, 7, 8, 9, t, e, 0.

ART. **54.** *To express a given number in any proposed scale*

Let N be the number, and r the radix of the scale

Then if a_0, a_1, a_2, &c., be the unknown digits

$$N=a_0+a_1r+a_2r^2+a_3r^3+, \text{\&c.}$$

If N be divided by r, the remainder is a_0,
if the quotient be divided by r, the rem. is a_1;
if this quotient be divided by r, the rem. is a_2,
and so on, till the last quotient is 0. The last remainder will evidently be the figure in the highest place.

Therefore, all the digits a_0, a_1, a_2, a_3, &c., are the successive remainders obtained by dividing the given number N, *and the successive quotients, by r the radix of the proposed scale.*

Ex. Transform 329 in the common scale, into the quinary scale whose radix is 5.

5	329			
5	65, 4 ; ∴	1st remainder		$a_0=4$,
5	13, 0	2d	"	$a_1=0$,
5	2, 3	3d	"	$a_2=3$
	0, 2	4th	"	$a_3=2$.

∴ the number required is 2304.

To verify this result we must have

$4+0\times5+3\times5^2+2\times5^3=329$, which is found to be correct.

By the same method a number may be transformed from any given scale to any other of which the radix is given. But in performing the division, it must be recollected that the radix is not 10, but some other number. In general, it is best to change the number to the denary scale, and then from that to the proposed scale.

Ex. Transform 3256 from a scale whose radix is 7 to a scale whose radix is 12.

Observing that the digits in 3256 increase from right to left in a *sevenfold* ratio, the division by 12 is performed thus,

12	3256		
12	166, 4 ; ∴	1st remainder	$a_0=4$,
12	11, 1	2d remainder	$a_1=1$,
	0, 8 ∴	3d remainder	$a_2=8$.

∴ the required number is 814.

Or thus, 3256 in the septenary scale is

$$6+5\times7+2\times7^2+3\times7^3=1168.$$

12)1168
12) 97, $4=a_0$,
12) 8, $1=a_1$. *Ans.* 814.
0, $8=a_2$.

Art. 55. *In any scale of Notation whose radix is r, a number* N *when divided by $r-1$, leaves the same remainder as the sum of its digits leaves when divided by $r-1$.*

For, let $N=a+br+cr^2+dr^3+$, &c.,

$=b(r-1)+c(r^2-1)+d(r^3-1)+$, &c.

$+a+b+c+d+$, &c.;

then since each of the factors $r-1$, r^2-1, r^3-1, &c., is divisible by $r-1$, (Algebra, Art. 83,) it follows that N and $a+b+c+d+$, &c., when divided by $r-1$, will leave the same remainder.

Cor. In the common scale of notation since $r=10$, $r-1=9$, therefore, *every number when divided by* 9 *will leave the same remainder as the sum of its digits when divided by* 9.

From this property is derived the rule for testing the accuracy of the operation of Multiplication, by *casting out the nines.*

Let A and B contain a and b nines respectively, with the remainders r and r', so that

$$A=9a+r, \quad B=9b+r';$$

$$\text{then } AB=(9a+r)(9b+r'),$$

$$=81ab+9br+9ar'+rr',$$

$$=9(9ab+br+ar')+rr';$$

$\therefore$ AB and rr', when divided by 9, leave the same remainder, that is, the sum of the digits of the product, when divided by 9, leaves the same remainder, as the sum of those of the product of the partial remainders leaves when divided by 9.

Ex. If $A=327$, and $B=248$; then $r=3$, and $r'=5$;

also, $AB=81096$, and $rr'=15$.

By casting the nines out of the sum of the digits in each of these products, we find the remainder in both cases is the same, that is 6.

NOTE.— This method fails to detect an error in either of the following cases: (1), when one or more ciphers have been omitted in the product; (2), when any of its digits are misplaced; and (3), when the error is equal to 9, or any multiple of 9.

EXAMPLES.

1 The number 4954 expressed in a different scale of notation becomes 20305; what is the radix of the scale?

$$20305=2\times r^4+0\times r^3+3\times r^2+0\times r+5=2r^4+3r^2+5;$$

$$\therefore 2r^4+3r^2+5=4954;$$

$$\text{whence } r^2=49, \text{ and } r=7.$$

2. Find the radix of the scale in which 95 is expressed by 137 *Ans.* 8.

3. Find the radix of the scale in which 803 is expressed by 30203. *Ans.* 4.

4. Find the radix of the scale in which the double of 145 is expressed by the same digits in the same order. *Ans.* 15.

5. Express the common No. 5381 in the ternary and nonary scales. *Ans.* 21101022, and 7338.

6. Express the quinary No. 34402 in the quaternary scale. *Ans.* 212231.

7. Express the common Nos. 6587 and 3907 in the duodenary scale; and then find their product.

Ans. 398*e* and 2317; product 8751215.

8. Multiply 24305 by 34120 in the senary scale. *Ans.* 1411103040.

9. Divide 95088918 by *tt*4 in the duodenary scale. *Ans.* *t*4*tee*.

10. Extract the square root of 25400544 in the senary scale. *Ans.* 4112.

11. Out of the series of weights of 1 lb., 2 lbs., 4 lbs., 8 lbs., &c., how many must be selected to weigh 153 lbs?

Solution. 153 must be expressed in the binary scale, and it is $10011001=2^7+2^4+2^3+1=128+16+8+1$; that is, we must take the weights 1 lb., 8 lbs., 16 lbs., and 128 lbs.

12. What weights of the series 1 lb., 3 lbs., 9 lbs., 27 lbs., &c. must be selected to weigh 1319 lbs.?

Solution. 1319 in the ternary scale is $1210212=3^6+2\times3^5+3^4+2\times3^2+3+2$;

$$\text{but } 2=3-1;\ \therefore\ 2\times3^5=3^5(3-1)=3^6-3^5;$$

$$3^6+3^6=2\times3^6=3^6(3-1)=3^7-3^6;$$

$$2\times3^2=3^2(3-1)=3^3-3^2,$$

$2=3-1$, and $3+2=3+3-1=2\times3-1, =3(3-1)-1=3^2-3-1$; hence the expression becomes

$$3^7-3^6-3^5+3^4+3^3-3-1=3^7+3^4+3^3-(3^6+3^5+3+1)$$

$$=2187+81+27-(729+243+3+1)=1319.$$

Hence, the three weights 2187, 81, and 27 must be placed in one scale, and the four weights 729, 243, 3, and 1 in the other.

13. Which of the weights 1 lb., 2 lbs., 4 lbs., 8 lbs., must be selected to weigh 1719 lbs?

Ans. 1 lb., 2 lbs., 4 lbs., 16 lbs., 32 lbs., 128 lbs., 512 lbs., 1024 lbs.

14. Which of the weights 1 lb., 3 lbs., 9 lbs., &c., must be selected to weigh 304 lbs?

Ans. Place 1 lb., 9 lbs., 81 lbs., and 243 lbs., in one scale, and 3 lbs., and 27 lbs., in the other.

ALGEBRAIC PARADOX.

It is shown, (Algebra, Art. 137,) that $\frac{0}{0}$ is the symbol of indetermination, and that it may represent *any quantity whatever.* A failure to observe this principle often leads to absurd conclusions; thus, if we take

thus, if we take	$a=x$;
multiplying by x	$ax=x^2$;
subtracting a^2	$ax-a^2=x^2-a^2$;
factoring	$a(x-a)=(x+a)(x-a)$;
dividing by $x-a$	$a=x+a$;
	or $a=2a$;
	or $1=2$.

Or thus,	$6+21=6+21$;
transposing,	$6-6=21-21$;
factoring,	$2(3-3)=7(3-3)$;
dividing by $3-3$	$2=7$.

In these, and other similar examples that might be given, the fallacy is caused by the indirect introduction of the indeterminate value $\frac{0}{0}$. Thus in the first example above, $a(x-a)=(x+a)(x-a)$, is the same as $\frac{a}{x+a}=\frac{x-a}{x-a}=\frac{0}{0}$ when $x=a$. Also, $2(3-3)=7(3-3)$ is the same as $\frac{2}{7}=\frac{0}{0}$.

Hence it is evident, that no reliance can be placed on conclusions derived from any process of reasoning where $\frac{0}{0}$ has been introduced.

THE END.

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